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MATHEMATICS SIMPLIFIED

AND

MADE ATTRACTIVE;

OR THE

LAWS OF MOTION

EXPLAINED.

BY THOMAS FISHER.

PHILADELPHIA:
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GREAT INDUSTRIAL EXHIBITION,

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AWARD OF THE JURY

PRESENTED TO THE ROYAL COMMISSION,

CLASS X.

PHILOSOPHICAL INSTRUMENTS, AND PROCESSES DEPENDING UPON THEIR USE.

JURY.

Sir DAVID BREWSTER, F. R. S., *Chairman*, Principal of the University of St. Andrews.
Sir JOHN HERSCHEL, Bart., F. R. S.
RICHARD POTTER, A. M. University College, London.
L. MATHIEU, France, Member of Bureau of Longitude, &c.
Baron ARMAND SEGUIER, France, Member of Institute, &c.
Professor COLLADON, Switzerland.
Professor SCHUBARTH, Prussia.
Professor HETSCH, Denmark,
E. R. LESLIE, United States, Artist, Member of the Royal Academy.
E. B. DENNISON, 42 Queen Ann Street.
W. H. MILLER, F. R. S. Professor of Mineralogy, Cambridge.
J. GLAISHER, F. R. S. Observer in Greenwich Observatory.
L. A. J. QUETELET, Belgium, Secretary to the Royal Academy at Brussels.
Rev. W. S. KINGSLEY, Sidney College, Cambridge.
Lord WROTTESLEY,
J. S. BOWERBANK, 3 Highbury Grove.

FISHER, (United States, No. 263, p. 1452,) has exhibited "Mathematics Simplified," consisting of some beautifully-drawn diagrams, intended to facilitate the study of Mathematics. His idea is that of teaching a physical geometry, either preliminary to, or when no better may be had, instead of the science. His

method of using the diagrams is by teaching each step by a course of reasoning and illustrating the laws by well drawn figures.

Nobody can question the great disadvantage under which students lie who have to apply geometry graphically, if their previous figures have been drawn only by hand; or what is worse, if badly drawn by ruler and compasses. It is doubtful which of the two is the greater evil—the giving a student ruler and compass as part of his course of geometry, or the making reasoning on badly drawn figures the only preparation for a draughtsman or architect, &c. This has been often said, but seldom accompanied with any proof of the satisfactory use which may be made of well drawn diagrams. Mr. Fisher has the merit of offering this proof to the Exhibition, with some ingenious ideas as to the manner in which the details may be managed. The attention he has paid to one point, viz: the exhibition of areas of given simple ratios under the same and different forms, is particularly beneficial. Some of his diagrams are, in this respect, excellent studies for an eye which is to be trained to correct estimation. His method is particularly applicable, for adults possessed of a power of thought, which requires to be enlisted and exercised to make their study agreeable, or even profitable.

Mr. Fisher's merit may be described as consisting in—1st, the application of the idea of teaching by physical perception, to a wider range of subject than merely making very exact drawings of the propositions to be demonstrated in Euclid; 2d, the ingenuity of his details; 3d, the beauty of the drawings.

The Jury considered Mr. Fisher deserving Honorable Mention.

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PREFACE.

It must be acknowledged that by the existing methods, the science of Mathematics is seldom effectively and comprehensively taught. Much time is wasted on it with comparatively little good result to the great majority of students. It is the purpose of the present work to show that very great improvements can be made in the methods of instruction.

It may be thought that Mathematics is the last subject in which important improvement is to be expected, and that even an apology needs to be offered for an attempt so novel and audacious. In the opinion of many persons, the very cobwebs which have gathered around this science have acquired a time-honored and venerable antiquity, and he must be a bold man who dares to stir them.

If the history of the past affords a presage of the present, a storm of opposition may be expected to any great improvement in this subject, similar to that which, in Italy, assailed Galileo's revelation of the Solar System, and in England, during many years, suppressed Harvey's demonstration of the circulation of the blood.

On the other hand, it is certain that the existing methods of teaching fail to interest the great majority of minds; that there are at this moment living, but few mathematicians in any comprehensive sense. However true, however ingenious, however worthy of immortal honor, the problems which the great Greek Geometers have bequeathed to us, as presented

to the student in the pages of Euclid, these pages are to the great majority, emphatically tedious and repulsive; they are very imperfectly comprehended, and therefore may practically be considered as little better than a sealed book; they are wanting in that simple, harmonious, and beautiful connection with each other, and with the glorious all-comprehensive superstructure of the physical laws, which would give them their highest value, and which has been reserved for a later age—an age subsequent to the labors of Galileo, of Tycho Brahe, and of Kepler.

Heretofore the appeal has been made so entirely to the memory rather than to the understanding, that we are not to be surprised at the repulsiveness of the methods by which the world has been so long mystified.

Mathematics, occupying as it does, a sort of intermediate position between physical science and mental philosophy, interwoven on the one hand with the forces and laws of Nature as well as of the Mechanic Arts, while, on the other hand as the science of absolute certainty, it is acknowledged to be the best means of cultivating a comprehensive and logical use of the intellectual faculties, is of indispensable importance to men of all professions; to the scholar and to the man of science, to the logician and to the mechanic, to the artist and to the engineer, as well as to the naturalist and to the metaphysician.

In Mathematics, the human intellect is justly considered as having gradually accomplished its most perfect achievement, revealing the noblest evidences of the wisdom, omnipotence and omnipresence of the Creator. The highways to its

grandest results have been discovered and investigated by men of the greatest genius—those noble martyrs of science, who, often amidst the neglect or persecution of their cotemporaries, have passed their lives in patient devotion to their favorite study, have gloriously surmounted its rugged acclivities, and thus have entitled themselves to immortal gratitude and honor.

Though the compilation of Euclid was well for the age in which he lived, though it was a good text book twenty centuries ago, it may now fairly be assumed that the progress of the human mind, so conspicuous in every other department of industrial pursuits, and of physical science, imperatively demands for the nineteenth century a more efficient development of this great subject.

The truths of Mathematics, the laws of Nature, have existed unchanged. Two hundred generations of men have lived and died, ignorant of these laws, to an extent not creditable to the intellectual activity of our race.

Any well sustained attempt therefore to improve this science, and to facilitate and insure its general acquisition, deserves attention. It is honorable, even to attempt to simplify, to explain and to make attractive, a subject which ought to be universally understood and universally popular.

The art of Printing is certainly capable of achieving, and the demands of the age imperiously require a much better, as well as a much wider education among the teachers of the people and the people themselves.

INTRODUCTION.

MATHEMATICS is a science in which nothing can be taken upon authority. It is emphatically a science of demonstration, where self evident conviction alone, in the mind of each and every individual, is relied upon to command and to compel involuntary assent.

What is demonstration? A demonstration is a chain of axioms. An axiom is a self evident or admitted truth, a stubborn fact that cannot be disputed. Self evident truths are simple results of observation, experiment and experience. Thus the apparently most difficult demonstrations rest upon the simplest truths.

"To Demonstrate, *verb active*, [*demonstro* Latin,] to prove with the highest degree of certainty; to prove in such a manner as reduces the contrary position to evident absurdity.

"Demonstration, *noun substantive*, [*demonstratio*, Latin,] the highest degree of deducible or argumental evidence; the strongest degree of proof; such proof as not only evinces the position proved to be true, but shows the contrary to be absurd and impossible." *Johnson's Folio Dic.* 1755.

Words are the signs of ideas. Things, on the contrary, are the sources of ideas. An idea is a mental image; an impression formed in the mind. An idea is an abstraction, when the source of it is not present. Thus, the idea or recollection of something that is past, is an abstraction. Thus, the general idea of justice, of truth, of natural philosophy, of mathematics, formed upon larger experience and education, is abstraction; it is generalization. But an abstraction not founded upon, and not consonant with Nature and Truth, would be a falsity, an insanity.

The child in his earliest progress in education, has every thing to learn by observation and experience. He learns not only that one finger and one finger make two fingers, but that one of any thing added to another similar thing, (one unit added to another unit) make two, &c. He thus has begun to acquire *abstract* ideas. An idea is a likeness. "Abstract" means, taken from. Every abstract idea must have been preceded by a reality submitted to his senses, and more especially to his sight. An abstraction is a likeness. A likeness of what? Of something you have previously seen or known.

Von Humboldt has said that no two persons assign precisely the same meaning to a word. The significance which is attached by each individual to any conventional language, is necessarily a result of his previous knowledge or education. All conventional language has for its base a natural language, a language of signs, a pantomimic language.

The meaning of the simplest word which a child learns, is a result of observation or of education. The Abbe Sicard, in his researches into the origin of language in connection with the means of teaching the deaf and dumb, has, I think, established this fact. The word *horse* must be associated with the actual animal; the word *kindness* must be associated with the natural expression of the emotion, before it can convey any meaning to the mind. The child first learns to discriminate a horse from a cow, and to associate the conventional words; he very gradually becomes, by experience, acquainted with a considerable number of animals, and thus is capable of making more general comparisons; in time he begins to observe their anatomy and comparative anatomy, and becomes an acute and philosophical observer. The idea of general classification at length occurs to the youthful Cuvier, and finally, in mature age, he writes his immortal work on the animal kingdom.

In like manner, the very simplest ideas of number, even that one and one make two, and that two and two make four, have to be acquired. A child, or an untaught savage may have a correct idea of numbers as far as he can count his fingers, but not beyond. It requires considerable familiarity with numbers to have an idea of hundreds, of thousands, or of millions, though we speak of them easily and thoughtlessly. Thus, the velocity of light, twelve millions of miles in one minute of time, though the fact that it is so, can be satisfactorily demonstrated, is too vast to be understood by uneducated persons without previous appropriate training.

In a science which professes to be so clear and certain as Mathematics, it is almost incredible that the use of terms should be so vague and unintelligible as we find it at this advanced period of the nineteenth century. There has evidently been too much assumption of profound meaning and too little honest intention to make intelligible to those of plain intellect and of common advantages, the truths of this noble and delightful science.

It is to this cause probably that a very large part of the difficulties of its acquirement is to be attributed, and that the nomenclature of the most certain of sciences is so little understood in an age like ours, boastfully claiming for itself an advanced intellectual and scientific culture.

Lord Brougham has lately said in the English Parliament, "that there is not a schoolmaster in England who has not himself need to go to school;" and though the remark may not be very flattering to the professors of the age, it is probably true in a much wider and more emphatic sense even than its author intended.

In this country we have a class of public men who make it their ambition to be considered friends of education, and who are always ready to flatter us with the idea that our system

of education is in a most flourishing condition. But with all this boasting it may be gravely doubted whether, relatively to the population, the number of men of really sound and comprehensive education is equal to what it was half a century ago.

Lord Bacon, in his much celebrated work, the "*Novum Organon*, or Rules for conducting the Understanding in search of Truth," published in 1620, Aphorism 90, says:

"In the customs and institutions of schools, universities, colleges, and the like conventions, destined for the seats of learned men, and the promotion of knowledge, all things are found opposite to the advancement of the sciences; for the readings and exercises are here so managed, that it cannot easily come into any one's mind to think of things out of the common road. Or if here and there one should venture to use a liberty of judging, he can only impose the task upon himself, without obtaining assistance from his fellows; and if he could dispense with this, he will still find his industry and resolution a great hindrance to the raising of his fortune. For the studies of men in such places are confined and pinned down to the writings of certain authors, from which, if any man happens to differ, he is presently reprehended as a disturber and innovator. But there is surely a great difference between arts and civil affairs; for the danger is not the same from new light, as from new commotions. In civil affairs, it is true, a change even for the better is suspected, through fear of disturbance; because these affairs depend upon authority, consent, reputation and opinion, and not upon demonstration. But arts and sciences should be like mines, resounding on all sides with new works and farther progress. And thus it ought to be, according to right reason; but the case, in fact, is quite otherwise; for the above-mentioned administration and policy of schools and universities, generally opposes and greatly prevents the improvement of the sciences."

Baden Powell, the present distinguished Professor of Mathematics in the university of Oxford, in his "*History of Natural Philosophy*" published in 1842, quotes and endorses this aphorism in the following paragraph:

"Bacon in proposing his new scheme of a philosophical institution, is of necessity led to justify the proposal by a reference to the services of existing institutions, and by showing how little they had done, or by their very nature and constitution were likely to do towards the real advancement of science. He censures, with severity indeed, but without bitterness, in strong terms, but with a masterly exposition of the facts, which evinces the perfect justice of his condemnation, the system of the colleges and universities of his day. [*Nov. Organon*, 1 *Aph.* 90.] He observes that the lectures and exercises were all of such a nature that no deviation from the established routine was likely to be thought of; that if a solitary attempt were made, the whole burden of it would rest on the individual, who would, moreover, find such attempts a serious impediment to his own advancement. The studies of these places were confined to a certain set of authors, as it were, imprisoned within those limits; any one who should venture to deviate from this course, would be immediately condemned as a turbulent innovator. But, he adds, in the arts and sciences, as in mines, the whole region ought to resound with new works and further advances. Such was the state of the case in his day; nor shall we find it much better in later times; for the present it must suffice to remark that, in regard to Oxford, when the scholastic forms were in great measure broken up under the reign of Cromwell, the favorable opportunity which was afforded for establishing a better system in their place was, as we have seen, not lost by a few ardent friends of true science; owing however to a variety of causes, their partial attempts failed, as far as the university

was concerned, and with the return of the Stuarts, the old system was re-established in all its authority."

Jared Sparks, on assuming the Presidency of Harvard University, June 20, 1849, says:

"Such have been the prodigious discoveries in the sciences, that the circle of what is called a liberal education has been greatly enlarged since the days of our fathers. The plan of instruction must be expanded in due proportions, and it is a problem difficult even in theory, but much more so in practice, how far one branch shall be curtailed to make way for another, which of them shall have the preference, and in what order the whole shall be so adjusted that each may fall into its appropriate place, and contribute its appropriate share. If you appeal to the general voice, you will be met by a conflict of opinions that would multiply rather than clear away the difficulties. Each man would judge according to his own ideas of the relative importance of particular studies; ideas derived rather from individual predilections, than from a comprehensive view of the whole subject. Some value lightly every kind of study which does not bear the stamp of direct utility. Others would have the thinking and reasoning faculties mainly cultivated; while others again would recommend, as the highest achievement, the art of communicating thoughts by a mastery over language, so as to captivate and convince a listening multitude, or impress great truths and brilliant conceptions on the reading world. * * * *

"There is so general an opinion among students, particularly in their early college life, that the mysteries of the Mathematics are above their comprehension, the mathematical faculty being, as they suppose, a gift of nature conferred on a few favored mortals only, that it might be hazardous to deny a position maintained by such numbers, although it may be doubted whether it is the result of resolute and laborious ex-

periment on their part. To accommodate such misgiving minds, however, it may not be amiss to leave Mathematics, at the proper stage of the college course, in the category of elective studies." * * * * * *

Of these two remarkable positions, the first would give the student to understand that the study of Mathematics is exceedingly mysterious, and only to be acquired by a few favored mortals; whilst the second consoles him by making this study a matter of mere option, and consequently of little or no importance.

The distinguished and learned gentleman having thus briefly and coldly dismissed the study of mathematics, proceeds to pass a very strong, just, and eloquent encomium on the value of the study of the ancient classics.

"For the last thousand years, at all times, and in all countries, where learning of any sort has been thought worthy of attention, under every form of government, and at every stage of civilization, the ancient classics have made a prominent part in every system of education. It is a significant fact, also, that the numerous colleges in our extended republic it is believed, have followed this example. * * *

"The highest attainment of a man of action and of thought is the art of using language with accuracy, elegance, force, and effect." * * * * * *

True! But how is the man of action and the man of thought to originate? How are boys to be trained, so as to become men of thought? is the question.

It must be admitted that the mere art of using language is not necessarily indicative of valuable thought. And there is also another side to this question, as it is often possible to teach acknowledged idiots to talk; as it is possible to teach persons of various grades of intellect above idiocy, to talk flippantly, fluently, and foolishly, so it is possible to

teach men of higher grades of intellect, and even of naturally excellent abilities, to make talking in an ostentatious and pedantic style a habit and an ambition, and thus immensely to overrate themselves, and their true value to society.

It ought to be taught, and generally acknowledged, that to listen well, to know when to listen, when to observe, and when to speak, is one of the highest accomplishments of a disciplined intellect. We want schools like that of Pythagoras, where students may go through a regular probationary course of silent listening and observation.

"The first step toward being wise is to know that we are ignorant." But most of us have not yet made even this first advance. Let us quote Locke, the greatest of mental philosophers:

"If we would speak of things as they are, we must allow that all the art of rhetoric, besides order and clearness, all the artificial and figurative application of words eloquence hath invented, are for nothing else but to insinuate wrong ideas, move the passions, and thereby mislead the judgment, and so, indeed, are a perfect cheat: and, therefore, however laudable or allowable oratory may render them in harangues, and popular addresses, they are certainly, in all discourses that pretend to inform and instruct, wholly to be avoided; and where truth and knowledge are concerned, cannot but be thought a great fault, either of the language, or person that makes use of them. What and how various they are, will be superfluous here to take notice: the books of rhetoric which abound in the world will instruct those who want to be informed; only I cannot but observe how little the preservation and improvement of truth and knowledge is the care and concern of mankind, since the arts of fallacy are endowed and preferred. 'Tis evident how much men love to deceive and be deceived, since rhetoric, that powerful instrument of error and deceit, has its established professors, is publicly taught,

and has always been had in great reputation; and I doubt not but it will be thought great boldness, if not brutality, in me to have said thus much against it."

It is one of the evils of our age and country, some parts of it especially, for talkers, men whose chief ambition it is to hear themselves talk, to usurp a position to which their thinking capacities are utterly disproportioned. The business of our legislative halls, our legal tribunals, our town meetings, is often impeded by this rhetorical verbiage, to an extent which demands abatement.

True eloquence is nature; it is the natural expression of thought, which art but attempts to counterfeit.

A skilful use of language, a comprehensive knowledge of things, and a wise and reflective sagacity in discriminating the relations of causes and effects, so far from being antagonistic or opposite, are in truth intimately subservient and essential to each other; and as an accomplished education is composed of an aggregate of many acquirements, whatever really facilitates, and thus saves time in the acquisition of one study, undoubtedly promotes the whole.

It is not only a false, but a very narrow and injurious doctrine, which supposes that a knowledge of those immortal discoveries of the Greek geometers, which form the basis of modern science, should excite in any degree less interest for those other forms in which Grecian intellect has embodied itself, for the veneration of posterity.

Would it not be absurd to suppose that a knowledge of the Greek language, which is undoubtedly the most brilliant achievement of human genius, and the most remarkable memorial of the earliest real civilization, should diminish the estimation of Greek sculpture or Greek architecture? yet this would not be more absurd than for this knowledge to diminish in the mind of one who understands them, the value of the geometrical discoveries of Pythagoras, "the wisest of the Greeks."

MATHEMATICS SIMPLIFIED.

CHAPTER I.

ON THE USEFULNESS AND IMPORTANCE OF MATHEMATICS.

THE usefulness of Mathematics is twofold. The first consists in its universal relation to physical science. The second in its relation to mental philosophy. It is perhaps equally valuable in both respects, as essential to every liberal and useful system of education.

It is important to men of every profession, as the means by which alone they are capable of understanding those principles which form the basis of all physical science—of all the physical laws of Nature. It may be said to form the framework of all the Sciences, and of all the Arts, or at least of all those of which Mechanics or Natural Philosophy constitute a part.

It is also, as the science in which our perceptions, reasonings, and conclusions are the clearest and least subject to doubt or dispute; that in which our most important mental habits are formed and exercised, and in the prosecution of which steady and continued thought is indispensable, acknowledged to be the best means of cultivating a logical and comprehensive use of our intellectual faculties. Mathematics, if it were taught intelligently, would be learned with delight by all persons who are ambitious of a good education, and in every country pretending to civilization, ought to form an essential element of popular education.

Instead, however, of extending his own views on the subject, the author begs to offer the following quotations from acknowledged authorities:

"Would you have a man *reason* well, you must use him to it betimes, exercise his mind in observing the connection of ideas, and follow them in train. Nothing does this better than mathematics, which, therefore, I think should be taught all those who have the time and opportunity, not so much to make them mathematicians, as to make them reasonable creatures."

[*Locke, on the Conduct of the Human Understanding, Introduction, Sec.* 6.

"It is to be observed that as there is no study which may be entered upon with a less stock of preparatory knowledge than mathematics, so there is none in which a greater number of uneducated men have raised themselves by their own exertions to distinction and eminence."—[*Dean Swift.*

"The mathematicians, from very plain and easy beginnings, by gentle degrees, and a continued chain of reasonings, proceed to the discovery and demonstration of truths that appear at first sight beyond human capacity." * * * *

[*Locke on the Improvement of our Knowledge, Bk.* 4, *ch.* 12, *s.* 7.

"In mathematics, and in natural philosophy, since mathematics was applied to it, we see the noblest instances of the force of the human mind, and of the sublime heights to which it may rise by cultivation."

[*Dugald Stewart, vol.* 3, *chap.* 1, *sec.* 2.

"I have mentioned *Mathematics* as a way to settle in the mind a habit of reasoning closely, and in train; not that I think it necessary that all men should be deep *mathematicians*, but that having got the way of reasoning which that study necessarily brings the mind to, they might be able to

transfer it to other parts of knowledge as they have occasion; for in all sorts of reasoning, every single argument should be managed as a mathematical demonstration, the connection and dependence of ideas should be followed, till the mind is brought to the source on which it bottoms, and observes the coherence all along."

[*Locke, Introduction to the conduct of the Human Understanding, Bk.* 4, *Sec.* 7.

"If a man's wit be wandering, let him study the mathematics; for in demonstrations, if his wit be called away ever so little, he must begin again."—[*Lord Bacon.*

"The habits of the mathematician afford exercise to that species of attention which enables us to follow long processes of reasoning, and to keep in view all the various steps of an investigation till we arrive at the conclusion. In mathematics such processes are much longer than in any other science, and hence the study of it is peculiarly calculated to strengthen the power of steady and concatenated thinking—a power which in all the pursuits of life, whether speculative or active, is one of the most valuable endowments we can possess."

[*Dugald Stewart, Part III, chap.* 1, *sec. III.*

The recognition of the importance of mathematics to medicine was cotemporaneous with the early labors of the founders of the science.

In the fourth century before Christ, Hippocrates, "the Father of Medicine," himself a distinguished mathematician, commended its study to his son, that he might more perfectly understand the situations, dislocations, contacts, and action of the bones.

In the next generation, Aristotle, amid various allusions to mathematics, particularises its connections with medicine.

Dr. Samuel Jackson, one of the most distinguished of American physicians, in a lecture introductory to a course on

the Institutes of Medicine, in the University of Pennsylvania, October 20, 1847, says:

"The branches of education the best adapted to prepare the mind for the study of medicine, in its new relations and character, are geometry, or the higher mathematics, physics, and chemistry.

"The objects of education should be to discipline the mind in the intellectual process by which it acquires knowledge; and in the methods of investigating and analyzing, by rigid scrutiny, the complexities of the compound phenomena or facts of science, by which it lays bare the truth; just as the brilliant gem is separated from the dross in which it is concealed.

"The spirit of geometry pervades all the physical sciences, and rules over all their operations and experiments. From the clearness, precision and order it imparts to mental operations, it may be looked upon as almost indispensable for the successful cultivation of science.

"He who possesses it is enabled to discover the fine thread, invisible to others, that leads through the mazes of the labyrinth that surrounds the shrine of truth and guides to its portals. He who has it not wastes his powers in fruitless efforts, or at best, meets with partial success in imperfect results. * * * * * * * * *

"The animal economy of man in its diversified relations and conditions, physiological and pathological, is an exceedingly complicated mechanism. * * * * * * A knowledge of physics and geometry, the ruling spirit of all scientific investigations, must lend the most material assistance in obtaining correct views of the structure of the animal economy, the nature of its operations, and the laws which preside over them. In this respect, this knowledge is of the greatest importance to the medical student.

Extract from a letter of Vice Admiral, Lord Collingwood, one of the ablest of seamen and naval chieftains, written after the battle of Trafalgar, to Lady Collingwood, on the education of their daughters.

OCEAN, JUNE 16, 1806.

* * * * * * * *

"How do the dear girls go on? I would have them taught geometry, which is of all sciences in the world the most entertaining: it expands the mind more to the knowledge of all things in nature, and better teaches to distinguish between truths, and such things as have the appearance of being truths, yet are not, than any other. * * * * * How would it enlarge their minds, if they could acquire a sufficient knowledge of mathematics and astronomy to give them an idea of the beauty and wonders of creation! I am persuaded that the generality of people, and particularly fine ladies, only adore God because they are told it is proper, and the fashion to go to church; but I would have my girls gain such knowledge of the works of the Creation, that they may have a fixed idea of the nature of that Being who could be the author of such a world."

CHAPTER II.

DEFINITIONS.

Locke, in his epistle to the reader, introductory to his Essay on the Conduct of the Human Understanding, says, "'tis ambition enough to be employed as an under labourer in clearing ground a little, and removing some of the rubbish that lies in the way to knowledge, which certainly had been very much more advanced in the world, if the endeavours of ingenious and industrious men had not been much cumbered with the learned but frivolous use of uncouth, affected, or unintelligible terms, introduced into the Sciences, and there made an Art of, to that degree, that philosophy, which is nothing but the true knowledge of things, was thought unfit, or incapable to be brought into well bred company, and polite conversation.

"Vague and insignificant forms of speech, and abuse of language, have so long passed for mysteries of science; and hard or misapplied words, with little or no meaning, have by prescription such a right to be mistaken for deep learning, and height of speculation, that it will not be easy to persuade either those who speak, or those who hear them, that they are but the covers of ignorance, and hindrance of true knowledge. To break in upon this sanctuary of vanity and ignorance, will be, I suppose, some service to the human understanding."

In any dictionary or cyclopedia extant, with which I am acquainted, the word *Mathematics* has not as it seems to me, a good, popular definition.

I have accordingly attempted to present one. But, in the first place, at the risk of being tedious, I have thought it just

to consult all of the existing definitions within my reach, and of these, to transcribe several which have appeared to me most valuable or peculiar, avoiding many which are evidently mere copies.

Mathematics, from the Greek noun μαθεμα [learning,] and the Greek verb μανθανω [to learn,] disco, intelligo, doceor, interpreto, sentio, imbuor, comperio, deprehendo, percipio, cognosco."—[*Constantine's Lexicon*, 1592.

"*Mathematics*, sciences or arts taught by demonstration, and comprehend four of the liberal sciences, Arithmetic, (wherein Algebra is comprehended) Geometry, Music, Astronomy."

[*Glossographia, or Dictionary of Hard Words, London, St. Paul's Church Yard*, 1656.]

With the ancients, mathematics meant all sorts of learning and discipline. To Plato, founder of the Greek school of mathematics, is attributed the earliest distinct discrimination of physical, mental, and moral science. Mathema, in the Greek vocabulary, more accurately meant physical science in all its amplifications, &c.

"*Mathematics*, the science of quantity; or a science that considers magnitudes either as computable or measurable."

"See Quantity and Magnitude."

"This word in its original μαθησις signifies *discipline* or *science* in the general, and seems to have been applied to the doctrine of quantity, either by way of eminence, or by reason of this having the start of all other sciences, the rest took the common name therefrom. See Science.

"*Pure* Mathematics consider quantity abstractedly, and without any relation to matter or bodies.

"*Mixed* Mathematics consider quantity as subsisting in material being: for example, length in a road, breadth in a river, height in a star or the like.

"Mixed Mathematics is Applied Mathematics. It is sufficient to determine an art to be a branch of mixed mathematics that pure mathematics are applicable thereto, *i. e.* that it may be explained and demonstrated from the principles of arithmetic and geometry. Such are *Mechanics*, which consider motion, or the laws of moving bodies. *Hydrostatics*, which consider the laws of fluids, or of bodies gravitating in fluids. *Pneumatics*, the air, with regard to the laws and mensuration thereof. *Hydraulics*, the motion of fluids. *Optics*, direct light or vision. *Catoptrics*, reflected vision—Reflection. *Dioptrics*, refracted vision—Refraction. *Perspective*, the images of objects, in order to delineate or represent them. *Astronomy*, the Universe, and the phenomena of the heavens. *Geography*, the Earth, both as in itself and in its affections. *Hydrography*, the sea, principally as navigable.—Navigation. *Chronology*, Time, with regard to the measuring and distinguishing thereof. *Gnomics*, or dialling, shadows in order for determining the hour of the day. *Military Architecture*, fortification the strength of places with regard to their defence against an enemy. *Civil Architecture*, building &c."

[*Chamber's Cyclopedia of Arts and Sciences, London*, 1743.

"Mathematics, *n. s.* *μαθηματικα* the science which contemplates whatever is capable of being numbered or measured; and it is either pure or mixed; pure considers abstracted quantity, without any relation to matter; mixed is interwoven with physical considerations."

[*Dr. Johnson's Folio Dictionary, London*, 1755.

"Mathematics, in general, is the *science of quantity*; or, the science which investigates the means of measuring quantity.

"Whatever is capable of increase or diminution, is called magnitude or quantity.

"Quantity is what is made up of parts, or is capable of being greater or less."—[*Maclaurin.*

"A sum of money, therefore, is a quantity, since we may increase it and diminish it. It is the same with a weight, and other things of this nature.

"From this definition it is evident that the different kinds of magnitude must be so various as to render it difficult to enumerate them; and that is the origin of the different branches of the Mathematics, each being employed on a particular kind of magnitude.

"Now, we cannot measure or determine any quantity except by considering some other quantity of the same kind as known, and pointing out their mutual relation. If it were proposed, for example, to determine the quantity of a sum of money, we should take some known coin, as a louis, a crown, a ducat, a dollar, or some other coin, and show how many of these pieces were contained in the given sum.

"In the same manner if it were proposed to determine the quantity of a weight, we should take a certain known weight, for example a pound, an ounce, and then show how many times one of these weights is contained in that which we are endeavoring to ascertain. If we wished to measure any length, or extension, we should make use of some known length, such as a foot."—[*Leonard Euler*, 1760.

"*Mathematics* treat of magnitude and numbers, instructing us how to measure, estimate and compute the different distances, magnitudes, and motions of bodies, with respect to one another."

[*Horne. State of the case between Newton and Hutchinson.*

"*Mathematics*, the science which investigates the consequences which are logically deducible from any given or admitted relations between magnitudes or numbers."

[*Brande's Encyclopedia of Science.*

"*Mathematics* the science of number and measure; the science of quantity; the science which treats of magnitude and number, or whatever can be numbered or measured."

[*Noah Webster, Hartford,* 1806.

"We are now able to define mathematical science with precision, by assigning to it as its object the *indirect* measurement of magnitudes, and by saying it constantly proposes *to determine certain magnitudes from others by means of the precise relations existing between them.*"

[*Auguste Comte's Philosophy of Mathematics.*

OUR OWN DEFINITION.

LET us ask ourselves, what is mathematics? It is the science of measurement. It measures quantities and numbers, magnitudes and multitudes. It teaches us to measure and to count.

Mathematics, the science of measurement, teaches us to measure accurately things of all sorts, of all sizes, of all forms. By mathematics we measure not only distances, lines, linear measurements,— surfaces, areas, superficial measurements,— spaces, masses, solid contents or cubical measurements; but also forces, motions, velocities, intensities, times, weights, and the various relations of all and each of these, respectively to each other, and to the simplest forms of measurement.

Mathematics uses numbers as one of its means of measuring. That part of mathematics which treats of numbers is called Arithmetic. Arithmetic is the use of numbers. Arithmetic is literally number measure. Algebra is a contrivance which records or represents concisely, mathematical truths. Algebra is in fact only a method of short hand. Algebra is to mathematics what stenography is to writing; a short hand method of recording what is already known.

Mathematics is a certain and unchangeable science.

the most comprehensive means of interpreting the immutable laws of Nature.

Mathematics begins with the simplest ideas of number and of measurement, which almost every child understands, and connects these simplest measurements like the links of a chain very naturally and beautifully with other measurements, which at first sight appear difficult; but if properly approached, the more advanced results can be reached with ease and pleasure, as a child by a graded pathway may delight to ascend a high mountain whose aspect from a distance was discouraging to the effort.

Mathematics, beginning with the simplest measurements, includes all measurements. It determines all the measurements used in the workshop of the mechanic, in the marketplace, and in the exchanges and commerce of nations. It comprises all adjustments of machines and mechanical calculations used in contriving, estimating, and constructing rude building materials into the most skilfully wrought houses or ships. Its applications include all labor-saving contrivances, from the most delicate cotton spindle to the colossal steamship. It defines the principles of drawing and perspective. On the one hand it makes the minuter microscopic measurements, and investigates the structure, crystalization, and organization of minerals, plants, and animals, while on the other, by the aid of the telescope (and the genius of the immortal Kepler,) it connects by simple laws of boundless comprehensiveness all terrestrial perspective,—all terrestrial mechanics, with astronomy or celestial mechanics, by which are determined with wonderful accuracy the distances, the times of revolution, the transits, the masses, and the comparative weights of the planets.

Mathematics, the science of measurement, is in ordinary language divided into

ARITHMETIC,—Number measure.

GEOMETRY,—Land measure.

ALGEBRA,—Letter or symbol calculation.

"*Arithmetic* is a science which explains the properties of numbers, and shows the method of computing by them."

[*Encyclopedia Brittannica, 7th edition.*

"Arithmetic, *n. s.* αριθμος [number,] and μετρέω [to measure,] the science of numbers; the art of computation.

"We have very little intelligence about the origin and invention of arithmetic; but probably it must have taken its rise from the introduction of commerce, and consequently be of Tyrian invention. From Asia it passed into Egypt, where it was greatly cultivated. From thence it was transmitted to the Greeks, who conveyed it to the Romans, with additional improvements. But from some treatises of the ancients, remaining on this subject, it appears that their Arithmetic was much inferior to that of the moderns."

[*Quoted from Chamber's Encyclopedia into Dr. Johnson's Folio Dictionary.*

Mr. Peacock, in his very elaborate history of Arithmetic, contained in the Encyclopedia Metropolitana, shows that very simple and practical methods of numeration, founded upon the most obvious scales of numeration, the fingers and toes of the hands and feet, have among all nations preceded the formation of numerical language.

In Arithmetic we have very many contrivances for simplifying and amplifying the use of numbers; of which a single example may suffice. In numeration, the value of a number depends not only upon the simple number for which it stands when alone, but upon the *place* in which it stands. Thus, in 888, the three eights mean eight, eight tens, and eight hundreds.

The place of a figure considered as affecting its value is

determined by the column in which it stands, and in the absence of succeeding figures to indicate the existence of other columns, their place is supplied by cyphers, which of themselves are considered as having no value. Thus, the 8 in 800 is of the same value as that in 863.

The early Greeks used the square, as the square of eight, the square of ten, the square of twelve, which term is borrowed from Geometry, and which well represents the multiplication table, as illustrated in the first Plate of this work, and which in fact is a very natural method of Arithmetic applied to Geometry. The invention of our common multiplication table is attributed to Pythagoras, who lived in the sixth century before Christ.

Arithmetic has for its primary divisions Addition, Subtraction, Multiplication, and Division. But Multiplication rightly defined is a wholesale addition. Division, in like manner may be said to be a short method of Subtraction.

"Multiplication rightly defined is a compendious addition."

[*W. Ludlam, St. John's College, Cambridge.*

The fundamental operations of arithmetic, as given in the Lilavati of the Hindoos, and as adopted in many of the books of arithmetic of the sixteenth century, are eight in number, viz: Addition, Subtraction, Multiplication, Division, the Square, Square root, Cube, Cube root. To the first four of these, the Arabs added two, viz: Duplation and Mediation, doubling and halving. It is quite possible, that, as a primary analysis of arithmetic, this is simpler, and more natural, and more easily understood than the very numerous rules of the moderns.

"*Geometry, n. s. γεωμετρία* [geometrie, French,] originally signifies the art of measuring the earth, or any distances or dimensions on or within it; but it is now used for the science of quantity, extension, or magnitude, abstractedly considered, without any regard to matter."

"Geometry very probably had its first rise in Egypt, where the Nile annually overflowing the country, and covering it with mud, obliged men to distinguish their lands one from another, by the consideration of their figure; and after which 'tis probable, to be able also to measure the quantity of it, and to know how to plot it, and lay it out again in its just dimensions, figure and proportion; after which it is likely a farther contemplation of those draughts and figures helped them to discover many excellent and wonderful properties belonging to them, which speculations were continually improving, and are still to this day. Geometry is usually divided into speculative and practical; the former of which contemplates and treats of the properties of continued quantity abstractedly, and the latter applies these speculations and theorems to use and practice, and to the benefit and advantage of mankind."

[*Quoted from Harris into Dr. Johnson's Folio Dic.* 1755.

"In the muscles alone, there seems to be more Geometry than in all the artificial engines in the world."

[*Ray on the Creation. Dr. Johnson*, 1755.

"*Geometry*, originally and properly the art of measuring the earth, or any distances or dimensions on it. But Geometry now denotes the science of magnitude in general, comprehending the doctrine and relations of whatever is susceptible of augmentation and diminution; as the mensuration of lines, surfaces, solids, velocity, weight &c. with their various relations."—[*Webster's Dictionary*, 1828.

Geometry, n. s. from γη [the Earth,] and μετρέω [to measure,] is the act of measuring the earth. This word originally meaning *land measure*, and embracing only a limited significance, had very improperly, as Plato thought, grown in his time into use as a substitute for the word μέτρησις which he very much preferred, and which meant measurement in its most comprehensive sense.

The meaning generally attached to the word geometry, in its modern use, not unfrequently includes this comprehensive signification, originally and properly attached to the general word mathematics.

Thus it is said that the principles of mechanics and of astronomy, (that is, of terrestrial and celestial mechanics,) are, purely considered, geometrical considerations; thus conic sections are geometrical considerations. In fact all the relations of length, breadth, and thickness, and of the forces, motions, or results which these represent are often said to be geometrical truths.

Thus as mathematics include multitude and magnitude—multitude as it is numerable, and magnitude as it is measurable; it may be defined to be the reciprocal comparison of multitude and magnitude. The modern use of the word geometry comprises all mathematics that is not arithmetic, and arithmetic is in truth only the application of numbers to measurement—number measure. "Measure," says Aristotle, "is that whereby quantity is known; but quantity is known as quantity, either by unity or by number." No geometrical argument is of force which agrees not exactly with an arithmetical calculus; and consequently all false reasonings in geometry may be examined and refuted most easily, certainly, and justly by arithmetical computation. Also all true conclusions and lawful demonstrations in geometry may be confirmed and illustrated by the help of an arithmetical calculus, which consideration being presupposed and established, the method of treating geometry will be found nowhere to differ from the calculations of arithmetic.

[*Barrow's Mathematical Lectures, Cambridge*, 1734.

A number is emphatically a measure. Numbers are as much the means and the subject of measure as quantities. This presupposes that the unit is known. Thus, one inch, one

mile, one pound, one ton, are units of measurement, and where their quantity is known as units, they give the power of counting, of computing, of measuring. If we know that the English ale gallon, usually taken as our water gallon, contains 282 cubic inches, and that one cubic foot contains very nearly $6\frac{15}{100}$ of such gallons—if we know the relations of these quantities as units, they become the means of estimating and determining any multiples of each other in any reservoir we may wish to measure.

Measurements after all are only comparisons; they denote nothing but that a magnitude is capable of being compared with other magnitudes, so that its quantity, however otherwise unknown and indeterminate, is capable of being known and determined from the relation which it obtains to them.

We require for measurement, standards mutually recognized. Thus, an Englishman who was ignorant of the French metre, would not derive any idea from it. He must translate or compare it with the English yard, or English inch, before he could have any accurate notion of the distance meant.

We have defined Mathematics to be the science of measurement. Measurement, in a comprehensive sense, including multitude and magnitude, force, motion, time, weight, &c. Measurements are either natural or artificial. A measurement, philosophically speaking, is only a comparison; it is a comparison with some known standard, either natural or conventional. Thus, a day, considered as the period of the earth's revolution on its axis, is an invariable natural measurement of time, the same in all ages and in all parts of the earth. We divide the day artificially into 24 hours of 60 minutes—1440 minutes, and 86,400 seconds. These are artificial, conventional measurements. We might divide the day into ten divisions, or any other number, but these would in any case be equally arbitrary sub-divisions of an invariable natural

standard, the day. A number of days taken together, form other conventional divisions, as a week, a fortnight, a month. The year, the period of the earth's revolution round the sun, causing the Seasons, is another natural standard of time. Thus, we make *days* the measure of the period of revolution round the sun of the nearer Planets, and *years* the measure of the more distant; the most distant, the latest discovered planet, Neptune, occupying many decades of years.

An inch, a foot, a yard, a bushel, a pound, are artificial measurements. We require these conventional measurements. We could not measure without them.

What is an inch? The United States being originally colonies of England, our measurements are those of the English standard. Thus, our inch is the English inch, one twelfth part of a foot, one thirty-sixth part of a yard.

The English yard was originally taken from the length of the arm of Henry I, King of England, and was afterwards confirmed by Acts of Parliament, from the reign of Henry III to George IV, in 1824.

The original yardstick is carefully preserved, we believe, in Guildhall, London, to compare others with, so as to have them all alike, and it is necessary to have officers whose duty it is to see that yardsticks and all other measures and weights are properly regulated.

There is an officer at Washington, whose duty it is to have correct weights and measures made for standards, and these are distributed to the different States and cities. There is also in most cities and large towns a Regulator of Weights and Measures, whose duty it is occasionally to examine the weights and measures in actual use.

The French metre is a sub-division of an invariable natural standard. It is one ten millionth part of the distance from the Equator to the Pole. It is equal to $39\frac{37}{1000}$ English inches.

Natural measurements are those not subject to variation from place, time, or chance, being the same invariably at all times, and in all parts of the earth. It is the purpose of the science of mathematics to contemplate all measurements whether natural or artificial, and all contrivances which have been devised for comparing the relations of these measurements with each other. Not only the relations of length, breadth, and thickness [distances, surfaces, masses] but also of forces, motions, velocities, intensities, times, weights, and the various proportions and comparisons of all and each of these, interchangeably with each other.

"In natural measurements we call to our assistance the presumed permanence of the great laws of nature with all experience in its favor, and the strong impression we have of the general composure and steadiness of every thing relating to the gigantic mass we inhabit—"the great globe itself." In its uniform rotation on its axis, accordingly, we find a standard of time, which nothing has ever given us reason to regard as subject to change, and which, compared with other periods which the revolutions of the planets about the sun afford, has demonstrably undergone none, since the earliest history.

"In the dimensions of the earth we find a natural unit of the measure of space, which possesses in perfection every quality that can be desired; and in its attraction combined with its rotation, the researches of dynamical science have enabled us through the medium of the pendulum, to obtain another invariable standard, more refined and less obvious, it is true, in its origin, but possessing a great advantage in its capability of ready verification, and therefore easily made to serve as a check on the other."

[*Sir John Herschel's Natural Philosophy.*

"Algebra, *n. s.* [an Arabic word of uncertain etymology;

derived, by some, from *geber*, the philosopher; by some from *gefr*, parchment; by others from *algehista*, a bone-setter; by *Menage* from *algiatarat*, the restitution of things broken.]

"This is a peculiar kind of Arithmetic, which takes the quantity sought, whether it be a number or a line, or any other quantity, as if it were granted, and, by means of one or more quantities given, proceeds by consequence, till the quantity at first only supposed to be known, or at least some power thereof, is found to be equal to some quantity or quantities which are known, and consequently itself is known.

"The origin of this art is very obscure. It was in use, however, among the Arabs, long before it came into this part of the world; and they are supposed to have borrowed it from the Persians, and the Persians from the inhabitants of India. * * * The first Greek author of *Algebra* was Diophantus, who, about the year 800, wrote thirteen books.—[*Dr. Johnson's Folio Dictionary.*

"Algebra is a branch of the mathematics, which has for its object whatever can be expressed by numbers, either exactly, or by approximation.

"In this respect, and also in its employing arbitrary signs to denote the things of which it treats, it agrees with arithmetic. The analogy between the two sciences induced Sir Isaac Newton to denominate it *Universal Arithmetic;* but by the application of algebra to geometry, the science has acquired a new character and new powers, which render this appellation too limited, and not sufficiently descriptive of its nature. In its present state, it is nearly alike related to arithmetic and geometry. In its application to both sciences, the reasoning is carried on by general symbols; its true character consists in this, that the results of its operations do not exhibit the individual values of the quantities which are the subject of investigation, such as we obtain in arithmetical or geometri-

cal constructions. They only indicate the operations, whether arithmetical or geometrical, which ought to be performed on the given quantities, to obtain the value of the quantities sought.

[*Encyclopedia Brittannica, 7th edition.*

"Algebra, Buchstaben-rechnung: letter calculation, algebraic reckoning; from Buchstaben, a type; letters applied to the designation of magnitudes."—[*German Dic.*

"Algebra, *n. s.* from the Arabic; the reduction of parts to a whole, or fractions to whole numbers; from the Arabic verb, which signifies to consolidate."—[*Webster's, Dic.* 1828.

"Analysis in mathematics, a method of solving or resolving mathematical problems.

"There are two general methods of finding truth in mathematics—Synthesis and Analysis. Synthesis is the composition, or the putting of several things together, (the building up as it were, of truth;) the method of Analysis consists more in judgment and readiness of apprehension, than in any particular rules. It furnishes the most perfect instances and examples of the art of reasoning. By it, geometrical demonstrations may be wonderfully abridged.

[*Chambers' Encyclopedia, London,* 1743.

"Analysis, *n.* [αναλυσις, of ανα and λύσις,] a loosing or resolving, from λύω, to loosen.

1. "The separation of a compound body into its constituent parts; a resolving, as an analysis of water, air or oil, to discover its elements.

2. "A consideration of any thing in its separate parts; an examination of the different parts of a subject, each separately, as the words which compose a sentence, the notes of a tune, or the simple propositions which enter into an argument. It is opposed to synthesis.

"In *mathematics*, analysis is the resolving of problems by

algebraic equations. The analysis of finite quantities is otherwise called Algebra or specious Arithmetic. The analysis of infinites, is the method of Fluxions, or the Differential calculus.

"In *logic*, analysis is the tracing of things to their source, and the resolving of knowledge into its original principles.

3. "A syllabus or table of the principle heads of a continued discourse, disposed in their natural order.

4. "A brief methodical illustration of the principles of a science. In this sense it is nearly synonymous with synopsis."

[*Noah Webster*, 1828.

"Analysis in *mathematics* is properly the method of resolving problems by means of algebraic equations, whence we often find that these two words, *analysis* and *algebra*, are used as synonymous.

"Analytic, or analytical, something that belongs to or partakes of the nature of analysis. The analytic method stands opposed to the synthetic."

[*Stewart's Philosophy, vol.* 2 *chap.* 4, *sec.* 3.

Algebra is a contrivance by which letters or symbols are used to represent quantities, and their relations to other quantities. Algebra is either Arithmetic or Geometry. It is applied to whatever can be otherwise stated as questions of Arithmetic or Geometry. It is a method of short hand, very convenient, and often nearly indispensable, as a means of recording the processes of Arithmetic and Geometry.

Algebra enables us to express a general truth, independent of any particular application of it. Sir Isaac Newton designated it "*Universal Arithmetic.*" But this does not seem a very satisfactory definition. We have many exercises in geometry and arithmetic thus expressed by algebraic symbols, and the utility and convenience of these symbols are undoubted, if used in their proper and true place. But the authors of

these short hand methods have been so numerous, and their symbolic language so various, that the ingenuous beginner, and even the ingenious student who has made considerable progress, is often hopelessly fatigued and mystified by them. Even so early as 1728, the publisher of the second corrected edition of Sir Isaac Newton's "*Algebra*, or *Universal Arithmetic*," (first published in 1707,) says: "It is true we have already a great many books of Algebra, and one might even furnish a moderate sized library purely with authors on that subject." But Sir Isaac's own methods are not models of perspicuity.

Prof. Barrow, in his mathematical lectures at Cambridge, 1734, page 28, (it may be remarked that he could not be ignorant of Sir Isaac Newton's work, published six years before,) says: "Perhaps some may wonder, that while I am endeavoring to make a perfect enumeration of all the parts (at least the principal parts) of Mathematics, I am wholly silent about that which is called Algebra, or the Analytic art. I answer that this was not done unadvisedly; because, indeed, *Analysis*, understood as intimating something distinct from the rules and propositions of geometry and Arithmetic, seems to belong no more to Mathematics than to Physics, Ethics, or any other science. No more is Synthesis, which is the manner of demonstrating theorems in contradistinction to Analysis."

Dugald Stewart, (vol. 2, chap. iii. sec. ii,) remarks that it may be questioned whether, among the ancient Greek geometers, this power, *the power of reasoning*, was not in a higher state of cultivation, in consequence of their ignorance of the algebraic symbols, than it exists in at this day, among the profoundest mathematicians of Europe.

The same distinguished philosopher devotes twenty pages, (vol. 2, chap. iv. sec. iii.) in which he most distinctly and formally dissents from Newton's statement of the priority and

importance of what are designated the analytic methods, and closes with the following paragraph:

"It were surely better that mathematicians should cease to give the sanction of their authority to this misapplication of the words analysis and synthesis, as it has an obvious tendency, beside the injustice which it involves to the inestimable remains of Greek geometry, to suggest a totally erroneous theory, with respect to the real grounds of the unrivalled and transcendent powers possessed by the modern *calculus*, when applied to the more complicated researches of physics."

The truth is, and ought to be admitted, that a synthetic process in Arithmetic or Geometry, equally with an analytical one may be at pleasure expressed by algebraic symbols.

Algebra is only short hand; as such, it is equally applicable to synthetical and to analytical processes. To speak of it, therefore, as an analytical method, in contradistinction to a synthetical, is both false and absurd.

CHAPTER III.

ON THE NATURE OF DEMONSTRATIVE EVIDENCE.

In the first chapter sufficient authorities have been quoted to establish the utility and the importance of mathematical studies as an indispensable branch of popular education. Mathematics is a science in which, however, authorities are of less importance than in any other. In mathematics, self-evident conviction, founded upon observation and experiment, is alone expected and relied upon to compel and command the assent of each individual. The opinions, or rather the researches of others have indeed great value to us, but only as they may be guides to the action of our own minds.

The present state of mathematical knowledge is a very slowly accumulated result of the labors of many contributors during a long series of ages. Among these are several men whose genius and patient industry have not been surpassed by the votaries of any other pursuit.

Yet, although the science is admitted to be the most remarkable monument of the human intellect, its methods of teaching, classification, and nomenclature appear to require revision, and it is the purpose of this chapter to offer some reason or apology for an attempt to simplify and to render attractive this great subject.

The proportion of students who succeed in acquiring a comprehensive knowledge of mathematics by the existing methods is surprisingly small, in any country. Our purpose is to consider whether the very limited number of persons who now derive benefit from its teaching is the result of difficulties inherent in the subject itself, or of the inadequate and ill-adapted methods of instruction which are generally adopted.

I shall here offer the opinions of others rather than extend my own observations.

I have before me "Observations on the Nature of Demonstrative Evidence, with an Explanation of certain difficulties occurring in the Elements of Geometry," by Thomas Beddoes, written at Oxford, and published St. Paul's Churchyard, London, 1793.

Its author, Dr. Thomas Beddoes, was Professor of Chemistry at Oxford, and subsequently was known as a distinguished chemist and physician in London, and contributor to medicine and chemistry, at a remarkable epoch of chemical discovery, when Black and Priestley were in the zenith of their labors. The work is dedicated to, and is sanctioned by the approbation of, Davies Gilbert, at that time Professor of Mathematics at Oxford, and subsequently President of the Royal Society. Prof. Gilbert was a gentleman of uncommon proficiency, and rare discernment in mathematical studies. It may be also remarked that Sir Humphrey Davy, perhaps the most distinguished chemist, and lecturer on Chemistry which England has produced, was a pupil of Dr. Beddoes, having been commended to him when a youth by Prof. Gilbert.

Dr. Beddoes had extraordinary merits as an original thinker. He very strongly approves of, and enforces the views of Locke. In his dedication he says:

"The more I consider the subject, the more I am inclined, in spite of Mr. Harris, to believe not only in the possibility, but the utility of rendering the elements of geometry palpable. If they be taught at an early age, a plan in which I think I see many advantages—models would make the study infinitely more engaging. From the mere slate and pencil most beginners experience a repulsive sensation. But if a child had something to handle, and to place in various postures, he might learn many properties of geometrical figures, without

any constraint upon his inclinations. He would have no difficulty in transferring the properties of palpable, to merely visible figures, nor in generalizing the inferences. *You* will not object that one cannot proceed far by this road: you will perceive that much more would be gained in reality than appearance. We should have laid a good foundation for the invaluable habit of accurate observation in general; and towards future progress in mathematics, we should have warded off the first disagreeable impression of the aspect of the science, which is so very apt to strike a damp to the heart of the beginner.

"I need not explain to you the advantage of trying to engage Fancy on our side by all the allurements we can offer to her. It is she that smooths every path, and strews it with flowers. We all, men and boys, follow with alacrity wherever she leads; neither the mind nor the body grudge any labor; and it is the enthusiasm she inspires that has worked so many miracles in art and science. By some strange fatality, however, she is neglected, if not affronted, in almost all the stages of education; and the first step in almost every species of instruction is to present knowledge to the student's imagination in conjunction with some melancholy and hateful accompaniment; which sort of management I conceive to have much the same kindly influence upon this faculty, as an unseasonable frost upon the tender petals of an expanding blossom.

"The mode of initiation in geometry which I propose, could not, unless I very much deceive myself, fail to render the impressions of sense more agreeable by rendering them more distinct. The *rigorously scientific* method, as it is supposed to be, seems, on the contrary, to aim only at rendering them as obscure as possible; an intention, I confess, perfectly in unison with the other parts of the established process of school and college stupefaction.

"Whether you will allow that this important point is likely to be in this manner attained, I am not sure. But you will agree with me in thinking that it is high time to discard Euclid's Elements. The science cannot be exhibited in a more disgusting form, as we may be convinced by daily instances. Nor are these Elements any way necessary to lay a good foundation in mathematics, for there are few I will venture to guess, of the eminent mathematicians of Europe that have been initiated by the study of Euclid.

"That, by laying Euclid aside, we should be deprived of what Bacon calls the intervenient advantages of mathematics, is to me a vain apprehension. Those who have dragged their understanding laboriously along the tiresome circuit of ancient demonstration, may be unwilling to grant that they have taken all these pains to no purpose. Yet they can hardly flatter themselves in secret that they have acquired habits of attention or abstraction superior to those who having pressed forward by the nearest road, enjoy both the direct and indirect profits of their labor.

"For want of time, or opportunity, or resolution, I am not able to take so comprehensive a view of the subject as I could wish, I cannot indeed suppose it possible that either branch of mathematics should ever cease to be '*Qualis ab incepto processerit*,' but I am apprehensive that I have not presented my observations in so advantageous a form as I might have done if I had possessed more knowledge. I hope, however, to be intelligible to those who may choose to examine them. In the main principle I cannot suppose myself mistaken; where I have committed errors in the application I shall receive correction with the cheerfulness becoming a person equally ready to hear and to tell the truth.

I am, with sincere regard, yours,

THOMAS BEDDOES."

Oxford, September 6, 1792.

We shall select a few detached portions, the heads of a few paragraphs, from the text, which may give some idea of its aim and purport.

Page 1. "In proportion as the writings of Mr. LOCKE rose in the public esteem, * *

Page 2. "Mr. LOCKE's success has, I imagine, already contributed, and will hereafter in a greater degree still contribute * * * * * * * one may, without impropriety, call a book popular that has gone through twenty editions.

Page 5. "In whatever study you are engaged, to leave difficulties behind is distressing; and when these difficulties occur at your very entrance upon a science professing to be so clear and certain as geometry, your feelings become still more uncomfortable, and you are dissatisfied with your own powers of comprehension.

Page 10. "The pretensions of the abstract sciences have, it must be confessed, something wonderfully alluring.

Page 13. "In mathematical reasoning, the mind grasps the conclusions with full assurance of their reality; we are satisfied that our advances in this science are actual acquisitions, and we find them as we go on continually capable of application.

"It may therefore be interesting to inquire into those circumstances which constitute the irresistible force of mathematical evidence. We shall, at the same time, if we are successful in this inquiry, discover upon what depends the difference in the cogency of proof between demonstrative evidence, and such evidence as less powerfully commands our assent. Without this, I do not see how we can ever take a clear survey of evidence in general, or enjoy the satisfaction of accounting to ourselves fully for our own conviction or belief.

"It seems to me, in the present state of our knowledge, so

easy to point out the nature of this and the other sorts of evidence, that I wonder how it can be mistaken. Yet frequently as the topic is expatiated upon, I know no book in which the true principles have been fully explained and applied; and in general, I have reason to believe that very erroneous ideas prevail upon a subject of unquestionable importance to the theory of the human understanding. * *

Page 15. "On examining a train of mathematical reasoning, we shall find that at every step we proceed upon the evidence of the senses; or, to express myself in different terms, I hope to be able to show *that the mathematical sciences are sciences of experiment and observation, founded solely upon the induction of particular facts, as much so as mechanics, astronomy, optics or chemistry.* In the kind of evidence there is no difference, for it originates from perception in all these cases alike, but mathematical experiments are more simple, and more perfectly within the grasp of our senses, and our perceptions of mathematical objects are clearer. * * * * * * *

Page 16. "No sooner do we look into an elementary treatise for the proofs of this opinion, than we meet with them at every step, in every demonstration; and I shall, I hope, be allowed to have established it firmly, if I shew that Euclid sets out from experiments, and proceeds onwards by their aid, appealing constantly to what we have already learned from the exercise of our senses, or may immediately learn. The same thing must needs be equally true of every other elementary author. After having exemplified the nature of demonstrative reasoning, I shall leave the reader to extend this mode of considering it to other cases, in a full persuasion that he will find the same process repeated in every demonstration upon which he may choose to make the trial. * * He here gives examples. * * * * *

Page 24. "I have been purposely prolix in this demonstration, to show how it begins in experiment, goes on by experiment, and ends in an experimental conclusion. There may be another use in insisting so particularly upon the nature of the reasoning process here: among those who teach mathematics without understanding their practical application, and also without entertaining a just idea of the nature of demonstration, there prevails a sort of pedantry, productive of infinite disgust to the learner. If by detached figures I could show the truth of any proposition in an instant, I am forbidden, because this is an *unmathematical* mode of proceeding; that is, mathematical reasoning is supposed to be something independent of experience, and the science to be more refined than the experimental sciences. Hence, if a Greek writer happens to have written a demonstration a mile long, which demonstration can be nothing but a concatenation of the results of observation and experiment, I must take this tedious round, rather than be allowed to arrive at the point desired by only traversing half a dozen yards, provided this shorter road leads through the unhallowed region of the senses. * *

Page 27. "In this manner does every demonstration proceed upon the results of experiments, as the reader will find, in as many instances as he shall take the pains to examine. And since the appeal in demonstrative reasoning is always made to what is now exhibited to the senses, or to what we have before learned by the exercise of the senses, too much pains cannot be taken at the commencement of the study of geometry, to satisfy the mind of the learner, by appealing to his senses. The more distinct and deep the impressions of sense are at the beginning, the greater will the power of *abstraction* afterwards be, when the progress of his studies shall have carried him into the higher mathematics. *Abstraction* is not in fact a distinct power, as the metaphysicians,

who seem to imagine that they increase the importance of their science, as they multiply distinctions, teach * *

Page 29. "By appealing in this manner to his senses, and making him feel the firmness of the ground on which he treads, one might probably instruct a boy, at an early age, in the elements of geometry, so as rarely to give him disgust, and frequently great satisfaction. He would, by imperceptible degrees, acquire the power of *abstraction*, or learn to reconsider each separate perception, as well as to combine them anew."

Dr. Beddoes next devotes a number of pages to the definitions and axioms used in Euclid, and other elementary books of geometry, and quotes Locke, Book IV. Chapter 7, Section 8, where Locke shows conclusively that these axioms are not the fundamental and primary truths first known to the mind, and that the other parts of our knowledge do not depend upon them, and also that these axioms are by no means the only self-evident truths, but that self-evident truths are in fact very numerous. He also shows that general ideas are formed slowly from the consideration of particular ideas with which we must be first familiar.

Page 38. "All accurate ideas founded upon measurement or careful comparison, must necessarily be subsequent to approximations suggested by a casual or distant survey of objects; and a good deal of perplexity, when persons first engage in the study of mathematics, arises from the result of vague observations, mixing itself with the result of the exact experiments employed in the reasoning of mathematics."

The boy knows that his hoop is round, and this aids in teaching him the distinct idea that in a circle all the lines drawn from the centre to the circumference are equal. If an existing circle, or what is supposed to be such, be found not to have the radii from the centre to the circumference

perfectly equal, it will in no manner affect the mathematician's reasoning on the properties of circles; it will only prove that the figure, or the wheel in question is not a circle according to the previous definition.

Page 58. "In Euclid's Elements, the truth seems to me to be so frequently obscured by demonstration, and so much disgust is often excited by his tedious method of proceeding, that were it not a violation of that loyalty which we owe to our masters, the Greeks, I wish the shortest possible method might be followed in teaching the rudiments of mathematics, by the help of simple satisfactory experiments. And if there be any one who should have learned geometry in this way, let him be assured that he holds his proficiency by a firm tenure. In this science there is no transcendental road; but I imagine a *royal* road might be struck out, though Euclid was of a different opinion. * * * *

Page 88. "Clear as it is, no use has I believe, yet been made of Mr. Locke's account of demonstrative evidence, either to solve difficulties or to improve the method of teaching in geometry: the shame, however, will be divided between so many culprits, and some of them so illustrious, that the share of each will be exceedingly small. For the subject has not fallen into the hands of ordinary compilers merely. Among the commentators upon Euclid, one might enumerate men of comprehensive views and various information; and authors, who aspire far beyond the commentator's highest praise, have formally discussed the nature of mathematical evidence. * *

Page 114. "Mathematics, in fine, teach either to measure or to count. The simplest, and the shortest way we can acquire either of these arts, the better, I believe, in all respects. We cannot possibly set about to learn either of them otherwise than by the use of the senses. And it is by no means impossible, that there may be a method of applying the

senses, in geometry at least, to far greater advantage than any practised at present; a method at once agreeable, expeditious, and calculated to invigorate every mental faculty. When such a plan of education shall be generally adopted, (and its outline is not extremely difficult to trace) that more knowledge than the learned and scientific now usually possess at forty, may be acquired by twenty, such a method of teaching geometry will probably form part of it." * *

CHAPTER IV.

EXPLANATION OF THE PLATES.

The science of measurement is capable of three great primary divisions. The first comprises measurements of one dimension only,—length. We can contemplate a length, a line, a distance of any given length. We can add to it; we can subtract from it; we can multiply its length till it may exceed the distance from the earth to the sun, or to the remotest of the planets; or any assignable length. Length is thus capable of almost infinite extension. We can also divide any given distance, any length, however great, and may continue to divide it, till it becomes exceedingly short, microscopically—mathematically small. But it is still *length* only. It is a line —a distance—a measurement of one dimension. It does not contemplate either breadth or thickness.

The second great division comprises measurements of two dimensions, length and breadth. These are planes—surfaces areas—superficial measurements only. Of this class of measurements the invariable law is the multiplication of the length by the breadth, the addition of the breadth to the length, (for multiplication is but a variety of addition,) which will in every case be true.

The third class of measurements are those of three dimensions, length, breadth, and thickness. Of these, the invariable law is multiplication, first of the length by the breadth, and secondly multiplying the product thus obtained by the third dimension, thickness, giving us cubical or solid measurements.

There is no fourth variety of measurement in nature, therefore every question of measurement, from the simplest to the most complex, must belong to one or to the other, or to combinations of these three. There can be no other.

PLATE No. 1.

THE purpose of this diagram is to exhibit in the most simple manner the three varieties of measurements (or dimensions) which exist in nature. It is important that these descriptions should be read with the diagrams in view. The student will be careful to examine each diagram step by step, as it is only in connection with those that precede, and with the whole as a series, that the purpose of any one diagram can be fully understood.

On the left. Figure 1; there are three lines of one inch in length, of differing breadth or thickness, but all of them equally lines, for a line may be indefinitely thin. There are also above these, similar lines of two, three, and four inches, and of twelve inches or one foot in length.

A line represents a distance, one dimension—length, for instance. It is not necessary that a line should be actually marked. A line run by a surveyor of land is usually marked by slicing the bark of trees adjacent to the line, but it is the corners which are chiefly relied on. A line of latitude, or the equinoctial line, where it crosses the earth or ocean, is measured from the heavens and the sun's apparent path; it is not traced upon the earth. A distance across a river, or an arm of the sea, is likewise accurately measured without the actual line, by triangulation.

Though we speak of a line, no line is actually drawn. It is an imaginary line, an air line, a line which may be measured, but it is not marked. In measuring a very short distance, as the width of a street or of a city lot, we may perhaps be said to draw a line when we stretch a tape across it. A line, as it represents length only, is often said to have no breadth; but a line as drawn upon paper if it had no breadth would not be visible.

A line, then, is a distance, one dimension; it may be from the earth to the sun, or to the most distant of the planets. We may conceive a line of any length, however great, and we may continue to add to it, or to multiply this length until it extends almost to infinity, (in other words, longer than any definite length.) We may then divide it, and may continue to divide it until it becomes indefinitely short, microscopically short, but it is still a line, however short the distance.

In Figure 2, we have an example of the second class of measurements, those of two dimensions, length and breadth. It is one inch in length, and one inch in breadth. We say once one is one. The square of one is one. One length multiplied by one breadth is one, (one unit of the second class of dimensions,) one area, one surface, one superficial measurement. We have also in Figure 3 two inches length by two in breadth, the area being the simple multiplication of the length by the breadth. In Figure 4 we have three inches by three inches, and in Figure 5 twelve inches by twelve inches, equal to 144 square inches, or one square foot. In every possible case of measurements of two dimensions, the rule is invariably the multiplication of the length by the breadth.

What is called the square root is the side of the square. It may be either the length or the breadth. The sign which is used to express the square root is $\surd$. Thus $\surd 4$ means the square root of 4; the square of 2 is 4, and the square root of 4 is 2; the square of 12 is 144, and the square root of 144 is 12. The square is expressed by 2, as A^2, which means length and breadth, or in arithmetic a number multiplied by itself.

Figure 6 represents an example of the third class of measurements, those having three dimensions, length, breadth, and thickness. Here we multiply, first, the length by the breadth; then, the product of the length and breadth by the third dimension, the thickness, giving solid measurements—the

cube. Once one is one, one length multiplied by one breadth is one, (one superficial or square inch.) Once one is again one, one length and one breadth is now multiplied by one thickness, (one cubic inch.)

In Figure 7 we have twice two are four, twice four are eight. Eight cubic inches is the cube of two.

In Figure 8 we have three times three are nine, three times nine are twenty-seven. Twenty-seven cubic inches is the cube of three.

To form the cubic foot from the square foot, as is represented in Figure 5, we multiply 144 by 12, giving 1728, the number of cubic inches in the cubic foot.

Great advantage to the beginner will be found to result from a tangible demonstration by handling a number, say as many as convenient, of cubic inch blocks of mahogany or any hard wood, and thus forming the cubes for himself.

The use of this diagram, as already stated, is simply to exemplify and point out length, breadth, and thickness, the primary elements of mathematics. To one or the other of these three relations, every possible mathematical question necessarily belongs.

We repeat, that every possible mathematical question, from the simplest which can be offered to the infant mind, to the most complex which can ever be devised by the ingenuity of man for the purpose of puzzling and of mystifying his fellows in exercises of calculation, must be founded upon one or the other, or upon combinations of these three elements. In Nature there is no fourth variety of measurement, no fourth power; in the language of arithmetic there are fourth, fifth, and higher powers, but these may be said to have no real existence; they have no application to forms or motions in Nature, and consequently are only mere exercises of calculation in Arithmetic or Algebra.

Perhaps the most important primary lesson which can be taught to the student of mathematics is the natural and direct connection between these three classes of measurements, with all the varieties and all the relations of motion. Every motion is a result of force. The nature or origin of force we cannot always understand; but we can observe and measure the motion, of which it is the cause.

One force produces motion in a straight line, and only in a straight line. Every motion which is in a straight line is a result of one single force, or it may be, of one aggregate force. It represents, and is capable of being measured by the first class of measurements.

A motion which is not in a straight line must always be a result of more than one force. Thus a stone thrown into the air returns to the earth. It has a curved linear motion. The one, (say the projectile,) force would have impelled it in a straight line, but gravity, a second force, acting upon it perpendicularly, brings it to the ground. In like manner, the movement of a stone attached to a string held in the hand describes a curved linear motion. So also do the revolutions of the planets round the sun, and of their satellites round the planets. These motions are in a plane. It has length and breadth. It is not one mere line. It possesses the properties of, and is capable of being measured by the second class of measurements.

The third class of motions are those of radiant phenomena, acting from a centre on all sides, as the diffusion of light, heat, gravity, electricity, sound, &c. All radiant phenomena illustrate each other. They diffuse themselves on all sides; they have length, breadth, and thickness. They belong to the third variety of measurements. As there are no more than three measurements of space and quantity in nature, so there are but three varieties and measurements of motion in the universe.

PLATE No. 2.

Every square and every parallelogram possesses the property of having its sides parallel to each other.

This Plate is designed to exemplify, First, the fact that every square and every parallelogram is capable of being divided diagonally into, and may therefore be considered as consisting of two similar and equal triangles.

Second. That this is true and invariable, not only of the right-angled parallelograms, but of those which are oblique.

Third. That the areas of triangles, like the areas of squares and of parallellograms, which are in fact only double triangles, are a simple result of the multiplication of their length by their breadth.

In Figure 1, we have an example of right-angled parallelograms of equal sides, that is, squares. Figures 2 and 3 are not right-angled; they are oblique and more oblique; and are called rhombs, but each of them may be divided into two equal triangles.

Figures 4, 5, and 6 are similar examples of parallelograms. Several diagonal lines are drawn, dividing these into equal and similar triangles.

Figures 7, 8, 9, 10, and 11, though the length and breadth are unequal, and the size and form and angles different, are further examples of this truth; which will be found on experiment to be invariable, and to apply without exception to every parallelogram, and the reverse, that by adding to any triangle a similar triangle of equal size, we should form a parallelogram, or double triangle, is obvious.

Figure 12 exhibits the truth that any triangle, if we double its length and breadth, quadruples its area, precisely the same as any parallelogram or double triangle. We may start from the left or the right, or from the apex, it does not alter the

fact. Figure 12 is at its top an angle of 120 degrees, or one third of 360 degrees, into which the circle is divided.

Figures 13, 14, 15, 16, are respectively at the top, angles of 90 degrees, 72, 60, and 45 degrees, ($\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{8}$, of 360;) they are successively more acute, but the increased acuteness does not alter the rule. It applies to every triangle, without exception.

Figure 17 is two triangles (of 30 degrees,) touching at the apex. On the right, the length and breadth are doubled, producing as in the previous examples, four similar triangles. On the left, the length and breadth is subdivided, and if, starting from the apex, or from either angle, we triple and quadruple the length and also the breadth, we would form successively, nine or sixteen similar and equal triangles, and so on. It is obvious, that if to the first triangle, or to any triangle whatever, we had before added another similar triangle (in reverse,) and thus made it a parallelogram, and had then quadrupled its length and breadth, we should have formed sixteen similar parallelograms or double triangles. The rule applies invariably and universally to all forms, as we must in due time make evident. It will be found to apply equally to all figures, whether bounded by straight lines, or by curved or crooked lines. It is the universal law of all superficial measurement to multiply the length by the breadth.

The phrase is a common one, to say that an area, a quantity, or a force, increases or diminishes as the "Squares of the Distances." This means simply and nothing more than the length multiplied by the breadth. It would often be equally true, and a more just and natural expression, to say that it increases as the circles of the distances, or as the triangles of the distances—or which is the same thing, as the length multiplied by the breadth of the distances. Thus, the left hand portion of Figure 17, starting from the apex, well illustrates the law of falling bodies—gravity, &c.

PLATE No. 3.

This diagram makes obvious to the eye the truth that all length and breadth justly considered are at right-angles with each other, and that if planes or solids are extended obliquely, drawn out, between the same parallels, the area is, the mass is, still invariable, however great the extension. For a familiar example, if you draw out a piece of india rubber, or of clay, or of dough, in length, it diminishes in breadth in the same proportion; in reality it is exactly the same quantity. This is perhaps, the simplest primary idea of fluent quantities, quantities flowing, quantities drawn out, water flowing, the relation of forces (and their result motions) to times, spaces, &c., in short, of many mathematical measurements, called by a variety of names; as Fluxions, Analysis of Infinities, &c.

Figure 1 is a parallelogram of 2 inches by 1 inch, say 2 inches length by 1 inch in breadth. As is exemplified in Figure 2, it is capable of division into two equal and similar triangles. In Figure 2 we have the diagonal line drawn, and the two triangles formed; and in addition a third similar triangle is drawn on the right of the parallelogram. The two triangles on each side of the central triangle are equal to it, and equal to each other. That on the left, together with the central one, forms a right-angled parallelogram; that on the right, together with the central one, forms an oblique paralellogram, obviously of equal area.

In Figure 3, we have again the same right-angled parallelogram as in No. 1, subdivided into four triangles. To this, on the right are added three triangles, which, with the one triangle which belongs to both, forms a parallelogram of the same area, and of twice the obliquity of that of Figure 2.

In Figure 4 we have again the same right-angled parallelogram as in No. 1, but subdivided into six triangles. To this

on the right are added five triangles, which, with the one triangle which is common to both, forms a parallelogram of the same area, but of triple the obliquity of No. 2.

In Figure 5 we have the same right-angled parallelogram, but subdivided into eight triangles. To this on the right are added seven triangles, forming with the one which is common to both, a parallelogram similar in area, but four times as oblique as No. 2.

In Figure 6 we have similar parallelograms of twelve subdivisions, six obliquities.

In Figure 7 we have parallelograms similar in area, but with 24 subdivisions, 12 obliquities.

In Figure 8 we have again parallelograms of the same area as Figure 1, with 30 subdivisions, 15 obliquities.

It must also be conceded that the principle thus exemplified or demonstrated is capable of any extension; as you extend a parallelogram or triangle, half parallelogram, obliquely between the same parallels, precisely as you increase the length, you proportionately diminish the breadth. The area will always be invariable and the same; if the extension in length were even hundreds, thousands, millions, millions of millions, indefinitely without limit, the breadth would be proportionately, indefinitely, infinitesimally diminished.

In Figures No. 9, 10, 11, the same right-angled parallelogram as in No. 1 is again drawn. The added triangles on the right represent various small and fractional degrees of obliquity, intermediate between No. 1, the right-angled parallelogram, and No. 2. But as long as the added triangle on the right, however small or fractional, or varying, is exactly equal to that which is cut off on the left, the area is absolutely the same, and these are only additional examples of the same truth.

Figure 12 exemplifies that what is shown in the preceding

figures to be true of parallelograms, is also true of half parallelograms or triangles. Figure 12 is an aggregate of triangles. The triangle on the extreme left, and that on the extreme right form together the right-angled parallelogram, Figure 1. The two next to these from the right and the left form the longest diagonal of the oblique parallelogram, No. 2. The two next from the right and the left form the oblique parallelogram, No. 3. The two next that of No. 4. The two next in like manner form No. 5; and thus the two triangles of corresponding obliquity represent, and their area is demonstrated by viewing them as combined into parallelograms. Thus the two central triangles correspond to the oblique parallelogram, Figure 8.

The parallelograms are all equal to each other, and contain the area of two square inches. The triangles, consequently, are all equal, and contain one square inch.

Figure 13 is another example of the same truth, as is shown in Figure 12. If we call from left to right *breadth*, the 32 triangles forming No. 12, are one inch broad, and two inches long; while the 16 triangles forming No. 13, are each two inches broad, and one inch long. The triangles in Figure 13 are of a different form, but equal in area, each being one square inch.

All the figures show that length and breadth justly considered are at right-angles with each other, and that area is the product of length multiplied by breadth.

PLATE No. 4.

This diagram consists of plane figures composed of several triangles. In Figure 1, starting at the centre, we have three similar and equal angles of 120 degrees, forming three triangles, which together form one triangle of equal sides, an

equilateral triangle. If we take separately one of these three triangles, and (as in Plate 2, Figure 12,) double its length and breadth, we quadruple its area. If we do the same to each of the other two triangles, we in like manner quadruple their area. If we now consider the triple triangle as one undivided whole, and double its length and breadth, we quadruple its area. What is true of each part separately, is true of the whole.

In Figure 2, we have at the centre four angles of 90 degrees, an aggregate of four triangles forming the well known figure called a square. If we double the length and breadth of each of these triangles separately, we obviously quadruple the area of each, and consequently that of the square they compose. The side of the inner square is equal to the base of one of the triangles, that of the outer square to the base of two. Having doubled the length and breadth, we quadruple the area.

In Figure 3 we have five triangles forming the regular pentagon or five cornered figure, illustrating the same principle. So in Figure 4 we have six triangles forming a six cornered figure, a hexagon; in Figure 5 eight similar triangles forming an eight cornered figure, an octagon; and in Figure 6 ten similar triangles forming the ten cornered figure, the decagon; each of these illustrating the same principle.

In Figure 7 we have on the left six triangles forming half of the twelve cornered figure, the dodecagon, and on the right twelve triangles forming half of the twenty-four cornered figure, the icosahedron. In Figure 8 we have on the left eight triangles forming half of the sixteen cornered figure, and on the right sixteen triangles forming half of the thirty-two cornered figure. All of these illustrate the same principle. The increasing number and acuteness of the triangles does not alter the rule; it applies to every triangle, however acute, and to every aggregate of triangles, however numerous.

These aggregates of triangles are frequently called by the general name of polygons, that is, many cornered figures. In Figure 1 they form a triangle within a circle. Here the segments, or interstices between the triangle and the circle, are large. As you increase the number of sides, as in Figures 2, 3, 4, 5, 6, 7, 8, the segments diminish, and the polygons increase, and make an approach towards the circle in a manner which can be determined and explained both by geometry and arithmetic.

This approximation towards the circle is capable of being continued, the interstices continually diminishing, and becoming less and less as you increase the number of sides of the polygon; but as there must always be a difference, an interstice, between the straight sides of the polygon and the curve line bounding the circle, the approximation can never be exhausted.

This subject, the approach of the polygon toward the circle will be farther illustrated in the explanation of Plate No. 5.

PLATE No. 5.

In Plate No. 4 we presented examples of a number of polygons drawn within the circle, and stated that the relation of their areas to each other, and to the circle enclosing them, (the circumscribing circle,) was capable of an arithmetical solution.

The relation of multitudes and magnitudes to each other are, almost in every case capable of an arithmetical solution. [Multitude is but magnitude expressed segregately. Magnitude is but multitude expressed aggregately.] A comprehensive knowledge of mathematics includes and comprehends all these relations. It also includes all and every thing which can be made the object of algebraic formulas, for

algebra is either arithmetic or geometry expressed in short hand.

In the notice of Plates No. 1 and 2, we asserted that every measurement of two dimensions was really, or was represented by, length and breadth—length multiplied by breadth, and that this law was invariable, and applicable to every form.

In Plate No. 5, we have a square. It is intended to be a square foot, but our remarks will apply equally to any square, without regard to size. It is divided by straight lines intersecting at the centre, into four equal squares. Each of these squares is again subdivided diagonally into two equal triangles. The four central triangles form another square of half the area of the first mentioned larger square, which comprises eight triangles, or four squares, as at first divided.

From the centre let a circle now be drawn, whose *radius* is six inches, whose *diameter* is one foot. The larger square outside comprising this circle is called the *circumscribing* square; the inside square is called the *inscribed* square. The last is obviously half of the first. But the purpose of this diagram as well as of No. 6, is to determine the area of the circle, and its relation both to the area of the inscribed, and to that of the circumscribed squares.

No other method has been, and it is believed no other method ever will or can be devised than that invented twenty-one centuries ago by Archimedes, and which he called the Method of Exhaustions.

This method consists in describing within the circle, based upon the inscribed square, a succession of right-angled triangles; and also in describing outside of the circle, commencing from the circumscribed square, a series of right-angled triangles; thus forming in combination with the squares, successively the octagon, the sixteen-sided, the thirty-

two sided figures, and so on, and approaching to the circle from the inscribed square within, and the circumscribed square without.

Now as the length and breadth, and consequent area of the inscribed and circumscribed squares respectively to each other is known, they form the means of demonstrating the length and breadth, and consequent area of these added triangles, and of the succession of resulting polygons, or many cornered figures constantly approaching to the circle, and enabling us with certainty to approach the true area of the circle. But still, though the intervening area can be continually diminished, it can never be entirely exhausted; for the reason that the curve line of the circle can never coincide exactly with the straight lines which form the polygon. There must always be an interstice between them.

The determining the areas of these right-angled triangles is dependant upon a knowledge of their length and breadth. It is dependent upon the truth of the great problem of Pythagoras, the 47th of the 1st book of Euclid's Elements, which is perhaps the most comprehensive, fundamental, and useful problem of geometry, and is capable of infinite applications.

The more matured demonstration of the application of this problem arithmetically and geometrically, may with the student's consent, perhaps better be deferred, or taken on his first reading partly on trust. It will be more completely understood on revision of this diagram after a full comprehension of the problem, which is illustrated in the 8th, and 11th, to the 19th, inclusive, Plates of this work.

We have repeatedly stated that length and breadth is the invariable law of all superficial figures; and this is indisputably true. But while all figures bounded by straight lines are capable of being compared with, and measured by each other with facility, all figures bounded by curve lines, as the circle,

and all varieties of the ellipse, can only be determined by approximation. The method of Exhaustions of Archimedes is an example of this.

To determine the area of the circle to its circumscribed square. If the radius be 1, the diameter will be 2, and the area of the circumscribed square, that is the square of the diameter, will be 4, and the area of the inscribed square will be exactly half of the area of the circumscribed square.

The areas of the circumscribed and inscribed squares being thus accurately measured, in order to determine the area of the circle it is simply necessary to calculate the ratio between these squares and the circumscribed and inscribed polygons of double the number of sides, that is eight sides. Then the same ratio for polygons of sixteen sides, then of thirty-two sides, &c. &c., until the series be carried to an infinite extent. Half of the sum of the differences of the areas of the two polygons added respectively to the inscribed, and deducted from the circumscribing polygon, will give an area as the intermediate, which will be an infinitely close approximation to the true area of the circle.

If, as in Plate 5, above, on the left, we first divide the side of the smaller or inscribed square into two equal parts, and, secondly, draw a line from the centre to the circumference of the circle, through the dividing point, and then draw lines from each corner of the square, meeting and touching the circle at this line, we shall form the octagon.

As the triangles thus cut out of the segment of the circle and added to the square in forming the octagon are right-angled, and as their length and breadth can be geometrically determined, their area is of course demonstrable.

Again, at the top of the figure proceeding to the right, we form in a similar manner from the octagon, the sixteen sided figure, inclosing other right-angled triangles, whose length, breadth, and area are demonstrable.

On the right of the Plate, from the sixteen sided figure we form the thirty-two sided figure, including other right-angled triangles, whose length, breadth, and consequent area are also capable of demonstration.

And thus again, from the thirty-two sided figure we form the sixty-four sided figure, which is as far as can be drawn distinctly on so small a scale as this diagram. It is obvious that a straight line and a curve line must, if placed side by side, always have an interstice between them, and therefore the straight sides of any polygon cannot ever be exactly commensurate or identical with the curve line of the circle, though these will continually approximate, and by a rapid ratio; and though we may proceed by decimals, arithmetically, and make approximations without limit in number, they can never exactly coincide with the circle.

For a familiar illustration of the possibility of a continued approximation without limit, I would present to each student ten thousand dollars, on the condition that he should make twenty successive divisions of the amount, halve it, and again halve it, and return the halves to me.

Thus, 1st, 10,000 divided by 2 = 5,000. 2d, 5,000 by 2 = 2,500. 3d, 2,500 by 2 = 1,250. 4th, 1,250 by 2 = 625. 5th, 625 by 2 = 312.50. 6th, 312.50 by 2 = $156\frac{25}{100}$. 7th, $156\frac{25}{100}$ by 2 = $78\frac{12}{100}$. 8th, $78\frac{12}{100}$ by 2 = $39\frac{6}{100}$. 9th, $39\frac{6}{100}$ by 2 = $19\frac{53}{100}$. 10th, $19\frac{53}{100}$ by 2 = $9\frac{76}{100}$. 11th, $9\frac{76}{100}$ by 2 = $4\frac{88}{100}$. 12th, $4\frac{88}{100}$ by 2 = $2\frac{44}{100}$. 13th, $2\frac{44}{100}$ by 2 = $1\frac{22}{100}$. 14th, $1\frac{22}{100}$ by 2 = $\frac{61}{100}$. 15th, 61 cents by 2 = $30\frac{1}{2}$. 16th, $30\frac{1}{2}$ by 2 = $15\frac{1}{4}$. 17th, $15\frac{1}{4}$ by 2 = $7\frac{5}{8}$. 18th, $7\frac{5}{8}$ by 2 = $3\frac{13}{16}$. 19th, $3\frac{13}{16}$ by 2 = $1\frac{15}{16}$, nearly. 20th, $1\frac{15}{16}$ by 2 = $\frac{31}{32}$ of a cent.

We have thus in twenty successive divisions cut down the amount to less than one cent, or less than one millionth of the original amount. $10,000 × 100 cts. = 1,000,000; but we might by adding six decimals again make twenty successive

divisions, reducing the amount to one millionth of one millionth, and again, by adding decimals, to an indefinite extent. But as our plan prescribes a continued division, the remainder is never exhausted. So that in theory, and in truth, the entire sum of ten thousand dollars would never be returned; there would always be a fractional remainder left.

The cent being the smallest in value of any coin in use, and the condition attached to my apparently very generous proposal, having diminished the gift to an amount less than one cent, it amounts practically to nothing at all. "I don't owe you a cent."

[A very good illustration, though not so obvious a one, is afforded by the air pump. In pumping the air from a receiver, the remaining air continues to expand itself without limit, and it is thus impossible completely to exhaust an air vessel. The scent of a single grain of musk diffuses itself for years.]

The application of this example of continued division applies beautifully, and with increased force to the geometrical exhaustion of Plates No. 5 and 6. It is obvious at once to every eye, that the portions between the inscribed square and the circle, and those between the circle and the circumscribing square, (the portions included and excluded, the integral and differential areas,) amount to more than one half; the ratio of approximation exceeds *three quarters*, the effect of which is to exhaust the integral and differential to less than one millionth in ten approximations, instead of twenty.

Say 1st, to $\frac{1}{4}$. 2d, $\frac{1}{4}$ of $\frac{1}{4} = \frac{1}{16}$. 3d, $\frac{1}{4}$ of $\frac{1}{16} = \frac{1}{64}$. 4th $\frac{1}{4}$ of $\frac{1}{64} = \frac{1}{256}$. 5th, $\frac{1}{4}$ of $\frac{1}{256} = \frac{1}{1024}$. 6th, $\frac{1}{4}$ of $\frac{1}{1024} = \frac{1}{4096}$. 7th, $\frac{1}{4}$ of $\frac{1}{4096} = \frac{1}{16000}$, nearly. 8th, $\frac{1}{4}$ of $\frac{1}{16000} = \frac{1}{64000}$. 9th, $\frac{1}{4}$ of $\frac{1}{64000} = \frac{1}{256000}$. 10th, $\frac{1}{4}$ $\frac{1}{256000} = \frac{1}{1000000}$. *Or, more accurately* suppose the radius of a circle to be denoted by unit or 1; the area of the circumscribing square will be expressed by 4, and the area of the inscribed square by 2.

Wherefore, the surface of the inscribed octagon is $= \sqrt{2} \times 4 = 2$ .8284271, and the surface of the circumscribing octagon is found by the analogy $2+2$ .8284278 : 2×2 : : 4 : 3 .3137085. Again, $\sqrt{(2$ .8284271 $\times 3$ .3137085$)=3$ .0614674, which expresses the area of the inscribed polygon of 16 sides, and 2 .8284271 $+3$ .0614674 : 2×2 .8284271, or 5 .8898945 : 5 .6568542 : : 3 .3137085 : 3 .1825979, which denotes the area of the circumscribing polygon of 16 sides. Pursuing this mode of calculation, by alternately extracting a square root, and finding a fourth proportional, the following Table will be formed, in which the numbers expressing the surfaces of the inscribed and circumscribed polygons continually approach each other, and consequently, to the measure of their intermediate circle.

NUMBER OF SIDES.	AREA OF INSCRIBED POLYGON.	AREA OF CIRCUMSCRIBING POLYGON.
4	2 .0000000	4 .0000000
8	2 .8284271	3 .3137085
16	3 .0614674	3 .1825979
32	3 .1214451	3 .1517249
64	3 .1365485	3 .1441184
128	3 .1403311	3 .1422236
256	3 .1412772	3 .1417504
512	3 .1415138	3 .1416321
1024	3 .1415729	3 .1416025
2048	3 .1415877	3 .1415951
4096	3 .1415914	3 .1415933
8192	3 .1415923	3 .1415928
16384	3 .1415925	3 .1415927
32768	3 .1415926	3 .1415926

What is called squaring the circle, is simply ascertaining the area of the circle relatively to that of the square.

PLATE No. 6.

Is a copy of Plate No. 5, with the addition that in Plate 5 the polygons based on the inscribed square only are exemplified. In No. 6, the polygons based on, and measured from the circumscribed square are added. The usual name for this process is the integral and differential calculus. It is used in approximating all curve lines, as the ellipses of every sort. A circle is a square cornered off. An ellipse is an oblong cornered off in a similar or analogous manner.

We have thus endeavored to give an idea of the relation which the area of the circle bears to that of the square.

PLATE No. 7.

We have more than once stated as an axiom, that the areas of all plane figures, of whatever form, whether bounded by straight lines or curved lines, are results of the length added to, or multiplied by the breadth.

The purpose of this Plate is to show that the relation of the area of any circle to the square including it, the circumscribed square, is an uniform one, whatever may be its size.

In this Plate, Figure 1, we have a square inch, within which is described a circle, and within the circle an octagon. The circle is the square cornered down to an uniform curve returning into itself. The area of the circle we have shown in the preceding page, and in Plates No. 5 and No. 6, is nearly as 31 .416, the square being 40 .000, or as 7 .854, the square being 10 .000. For the present purpose it is only necessary to make self-evident that the relation of any circle to the square circumscribing it is invariably similar.

In Figure 2 we have doubled the length and breadth of the square, and consequently the length and breadth of the

inscribed circle and octagon; and have quadrupled their area. In Figure 3 we again double the length and breadth of Figure 2. In Figure 4 we double the length and breadth of Figure 3. In each case we have quadrupled the area. We have thus in Figure 3, 4 inches × 4 inches = 16 inches area, and in Figure 4, 8 × 8 = 64 inches area, or 64 times the area of Figure 1; and the corresponding circle and octagon are 64 times the area of those in Figure 1.

In Figure 5, for further illustration, we have combined Figures 1, 2, 3, 4. It is obvious that the areas of each and all the constituent parts of these figures bear an uniform ratio to each other; not only the squares, octagons, triangles, and circles, but also the integral segments between the octagons and the circles, and the differential portions which form the corners between the circles and the squares. The length and breadth of the square being successively doubled, and the area quadrupled, all of the constituent parts which compose the square are increased in the same proportion—in the same ratio. The usual axiom is, all circles are to each other as the squares of their diameters. All plane figures are to each other as the squares of their homologous sides. [Homologous is a Greek word signifying similarly described.] But there would be equal truth and propriety in saying or teaching as an axiom, all squares are to each other as the circles of their diameters. All plane figures are to each other as the circles of their homologous sides. The real truth is that all plane figures, whatever may be their outline, regular or irregular, bounded by straight lines or bounded by curve lines, whether squares, octagons, or triangles, whether circles or ovals, or any irregular figure, are to each other as their length multiplied by their breadth.

We have thus in this diagram established the circle of one inch in diameter, of one inch in length, and one inch in

breadth, as an Unit Circle, which bears the same ratio to the other circles of increased length and breadth that the respective squares do to the unit square of one inch.

In Figure 6, omitting the squares, we have at the centre simply the circle of one inch in diameter, and successively circles of two, three, four, five, six, seven, and eight inches in diameter. These diameters of course represent length and breadth, and multiplied, each into itself, give areas of one, four, nine, sixteen, twenty-five, thirty-six, forty-nine, sixty-four, for the areas of the respective circles.

PLATE No. 8.

The fine geometrical lines on which this figure is constructed, were drawn originally with all practicable accuracy. The stronger black and red lines are drawn within, and entirely independent of these, and are solely for a more distinct impression on the eye. The coloring which I have thought it proper to use must be regarded as an expedient independent of the geometrical value of the diagram, and entirely defensible only on the plea of greater distinctness.

Drawn in black from the centre, we have the square inch, and the circle of two inches diameter, whose radius is the side of this square. Above, and parallel with this on the left, we have the square of two inches sides and its corresponding circle. On the right, occupying a similar relation to the centre, we have the square of four inches sides, and that of eight inches sides, with their respective circles.

Drawn in red, from the centre on the diagonal of the first square inch, we form the second square, which (by the great problem of Pythagoras, the 47th of the first book of Euclid,) doubles the area of the square inch. It is our first demonstration, and probably the simplest which can possibly be given of

this invariable problem, *that the square drawn upon the longest side, (that is the hypothenuse,) of any right-angled triangle is equal to the sum of the squares (which may be drawn) on the other two sides.* The squares of the two sides are faintly drawn, and each side being one inch, these squares are square inches, and of course equal to the original square inch. If we now consider square No. 2 as divided diagonally into four equal triangles, one of these triangles will be the half of the square No. 1, and square No. 2 therefore obviously doubles the area of square No. 1.

Parallel with square No. 2, the square No. 4, and the square No. 6, both drawn in red from the centre on the diagonal of squares No. 3 and No. 5, repeat the same problem. We have therefore the four black squares, (with their respective circles,) the relation of which, each to the other, we can demonstrate by drawing two lines parallel with the sides of the larger square, intersecting it at its centre, dividing it into four squares, each equal in area to the smaller. And also the three red squares with their respective circles, the relation of which to each other can be demonstrated in the same manner, by drawing two lines parallel with the sides of the larger square, intersecting at its centre. While at the same time the relation of the black squares to the red squares is alternately and successively capable of demonstration by diagonal division into four triangles, intersecting at the centre of the square, forming examples of the problem of Pythagoras.

The relation of the squares and the circles being in every case similar, they respectively double each other's areas as you build up from the square inch, and diminish by a similar ratio in reversing the process.

The analogy between this diagram and No. 7 is perfect, although in arrangement they are different. In No. 7, the unit circle has the side of the square inch for its diameter.

In this diagram the smallest or unit circle has the square inch for its half diameter or radius. The areas of these circles are obviously quadruple those in No. 7, but their ratio to each other is precisely similar.

We may start at the centre, that is at the one square inch, or the triangle, which is one half of this square, and may demonstrate the area synthetically, by addition, or we may start from the largest square of eight inches sides, and proceed by division, analytically, towards the smaller areas. There is no difference in principle, synthesis is but addition or multiplication, analysis is but subtraction or division.

They double each other's areas successively, the ratio of the squares and circles remaining unchanged.

Whenever a figure is bounded by straight lines, its area is capable of most prompt and definite measurement. Where a figure is bounded by curve lines, its area, compared with figures bounded by straight lines must be ascertained by approximation. By using a curve line figure for our unit of measurement, as a circle of one inch in diameter, we can measure similar curved line figures with greater simplicity; thus, we can definitely measure the relation of circles to each other by simply multiplying their diameters, that is, their length and breadth; but comparing them with squares or right lined figures is another thing, and this must be done by approximation.

THE APPLICATION OR USE OF PLATE NO. 8.

The distinction between pure and applied mathematics as usually made is, in our judgment, vague, and thus is productive of great uncertainty to the learner, unless especial care is taken to make a true definition of both. And if this is done, there will be found to be in principle no difference whatever, and none, in fact, but what is inseparable from the imperfect application of human measurements and human mechanism.

It is true, that in applied mathematics we often have to take into consideration some physical fact, inaccuracy, impediment, or source of error, which alters the exact mathematical result; but in this case we always have reason to attribute the error to some oversight, disturbance or irregularity resulting from our own incomplete comprehension, or from imperfect application, and not to any defect in the principle itself.

In Euclid we have first, the book called Geometry, and secondly Trigonometry, and subsequently ten other books, in which proportion, &c. are treated. The successors of Euclid have added three more books, so that we have fifteen books of "Pure Mathematics." What is pure mathematics? Let us consider.

The instant we begin to make any application of mathematical principles to mechanics or to natural philosophy, we are told that is *not* pure mathematics. Then what is pure mathematics? Pure mathematics, we are given to understand, is Geometry and Trigonometry, and the whole twelve books of Euclid. But the figures drawn on the paper in exemplification of the problems are not pure mathematics, for they are subject to the inaccuracy of the drawing. If you say that two and two make four, this is pure mathematics. If you say that ten yards and ten yards are twenty yards, or that ten yards in length and ten yards in breadth are one hundred square yards in area, these are pure mathematics.

But if you come to make the actual measurement, you are subject to some inaccuracy, which, as already stated, is inseparable from all human measurements, and human mechanism. In mechanics you have some friction. In astronomical observations you are subject to more or less mist in the atmosphere, and you have to make allowance for refraction.

The necessary allowance for these being made, pure mathematics and mixed mathematics assimilate; they are the

same thing. Pure mathematics is the mathematical truth as it exists in your mind.

The perfect intellectual conception of a law of Nature is pure mathematics. But in Nature the combined action of various laws modifies the application of such a law in particular instances, and this must be allowed for.

In applied mathematics, therefore, we approach to absolute truth as nearly as we can; but we have to make allowance for our own errors of observation, for imperfection of our instruments, and for disturbing or contravening causes in Nature; and, therefore, though we know that we are correct, and can prove it, we always admit the universally acknowledged liability to some inaccuracy. It is thus that the most skilful accountant never omits to write "errors excepted."

It is in astronomy, however, that applied mathematics approaches most nearly to absolute truth. It is here that the accurate accordance of our measurements of the revolutions of our sister planets, with those diurnal and annual revolutions of the orb we inhabit, by which we measure terrestrial time, and witness the sure return of the seasons recorded among the stars: enables us to compare them, and to predict with a precision almost identical with absolute truth the transit of a planet, and the occurrence of eclipses, over long intervening centuries.

We come now to consider Plate No. 8 as a contrivance for illustrating, explaining, or demonstrating mathematical principles. We will first apply it to mechanics, and secondly to other subjects of natural philosophy.

Let us now consider Plate No. 8, as representing a wheel. The vertical line from the centre, eight inches in length, may represent a radius, a spoke, a lever of eight inches, for a wheel is an aggregate of levers.

If we apply a force at the end of the lever, the centre being

the axis or fulcrum, we necessarily exert our force in the circumference of the largest circle, say to the right. If our power is applied at the circumference of the smallest circle, one inch beyond the fulcrum, it will be exerted through one eighth of the distance, with one eighth of the velocity, but with eight times the power.

If our power is applied two inches beyond the fulcrum, it will be exerted in the circumference of the third circle of two inches radius, through one quarter the distance, with one quarter the velocity, but with four times the power.

If our power is applied four inches beyond the fulcrum, it will be exerted in the circumference of the fifth circle of four inches radius, through one half the distance, with one half the velocity, but with twice the power of the force exerted.

If our lever or radius were extended to the same length on the opposite side, we should apply our power in the circumference of the same largest circle, and of course through the same distance, with the same velocity, and with the same power. It would be a simple balance.

It is obvious, that if instead of the longer end of the lever, we were to apply the force at the other or shorter end of the lever, we should exert our power in the circumference of the larger circles with increased velocity, and through increased distances, but with proportionally diminished power. It is an exemplification of the truth that what is gained in time is lost in power, and the reverse.

This diagram also well exemplifies what is called in the books the second form of the lever. In this case, the lever terminates at the fulcrum, that is, at the centre. The weight or resistance is now between the fulcrum and the *hand*, and it is, consequently, only on our own side of the fulcrum that such a lever can act, and the direction of the power, as well as that of the weight or resistance is the same, or rather *in a curvilinear parallel with* the force applied.

It also exemplifies the third form of the lever as it is called, where the force is applied between the fulcrum and the resistance. This is, in truth, only the reverse of the second form of the lever. For illustration: Let the black line of eight inches represent a ladder of a given length, lying on the ground; let the ground at the centre be the fulcrum. If a force is now applied at one inch from the centre, and exerted through a given distance, in the circumference of the first circle, it will raise the top of the ladder through eight times this distance. But it will require eight times the force. Eight times the force exerted through one distance, is thus equivalent to one force exerted through eight distances.

The diagram also well illustrates the bent lever, or the crank motion. The bent lever does not alter the principle. It is obviously not different in principle from the straight lever. Bending it very little or bending it very much makes no difference. Its relative force is measured by the distance from the centre, or fulcrum, in either and every case.

The diagram also exemplifies the wheel and axle, the capstan and the pulley; the wheel is but an aggregate of levers, a perpetual lever. If we apply our force in the circumference of the largest circle, and wind up our rope upon the smallest circle, we shall wind up the rope through only one eighth of the distance in which the force is exerted, but we shall wind it up with eight times the power.

There is no form of the lever which this diagram does not represent. Thus much for mechanics.

Plate No. 8 also well exemplifies the balance and the steelyard.

Let the black line vertical from the centre, eight inches in length be one arm of a balance, and let us suppose the opposite arm to be extended equally on the opposite side. Let the centre be the fulcrum, and let the right side of the plate be

the top. One weight, say one pound weight, at eight inches from the fulcrum balances a similar weight at a similar distance opposite; and the vibration of the weights on either side will be through equal distances, and in the circumference of the same largest circle. It is obvious that this would represent the simplest form of the balance. Any similar weights at similar distances will of course balance each other.

But the one pound weight at eight inches distance may also be balanced by two pounds at four inches from the fulcrum, vibrating in the circumference of the circle of four inches radius, through one half the distance described by the one pound.

The one pound at eight inches may also be balanced by four pounds at two inches.distance from the fulcrum, vibrating in the circumference of the circle of two inches radius, through one quarter of the distance described by the one pound.

And the one pound at eight inches may again be balanced by eight pounds at one inch from the fulcrum, vibrating in the circumference of the circle of one inch radius, through one eighth of the distance described by the one pound at eight inches radius. It is plain that this is a simple linear measurement, and that distance and weight represent each other.

Thus: One pound weight, or any one weight at eight distances - - - - - - - - - - = 8

Eight weights, say eight pounds weight, at one distance - - - - - - - - - - = 8

Also: Two weights, say two tons, at four distances, = 8

Four tons at two distances, - - - - - = 8

There might be any other representatives of distance and of weight which, multiplied into each other would give the result, 8, which of course would interchangeably balance with any of these, and with each other.

Sixteen pounds weight, at one half a distance, - - = 8
One half of one pound weight, at sixteen distances, - = 8
Thirty-two pounds, at one quarter of a distance, - = 8
One quarter of one pound, at thirty-two distances, - = 8
Also $\sqrt{8}$ pounds at $\sqrt{8}$ distances - - - - = 8

Plate No. 8 also well illustrates Natural Philosophy. What is Natural Philosophy? It is the laws of certain natural phenomena, the most remarkable of which are heat, light, sound, magnetism, electricity, galvanism, gravitation.

Let our first illustration be of light. Let the centre of the diagram be the pupil of the eye. If an object measured, let us suppose from the vertical line to the right, at the distance of one inch from the centre, occupy one eighth of the circumference, or 45 degrees of the smallest circle, the same object, at eight times the distance, at the circumference of the largest circle, would occupy only an eighth of one eighth, or five and five eighth degrees of this circumference. This is perspective. This illustration is on a small scale; but if instead of one inch we commence with a foot, or a yard, or a mile, the principle would be the same.

In further exemplification, let us suppose ourselves on the summit of a very high mountain on a fine day; the extent of our vision might be such, that the one inch might represent 10 miles, the two inches 20 miles, the four inches 40 miles, the eight inches 80 miles; every object, from the bush at our feet in the immediate foreground, to the most distant object in the extended landscape, would occupy in our field of vision its true mathematical relation to other objects, according to its distance in terrestrial perspective.

If we extend our vision beyond the earth—to the heavens, the same principle will be found to govern celestial perspective. Parallax is but another word for the same thing. Parallax is but the perspective of the heavens. We divide our

circumference into 360 equal parts, called degrees. Thus when we say the moon's diameter is a little more than half of a degree, we mean that it would take about 180 moon's diameters to reach from the zenith to the horizon, or about four times this number, 720, to form a continuous ring of moons encircling the whole horizon.

Now it is obvious, that if the moon were nearer to us, say only one eighth of its present distance, it would require only one eighth of the number of moons to form the entire ring. The moon would then have eight times its present apparent diameter, and the earth would receive from the moon 64 times as much of the moon's reflected light as it does at present.

We have already several times used the word angle; and as a good definition of it, may be useful, we attempt to supply one.

Angle, n. s. [angle, Fr. angulus, Latin.] The space intercepted between two lines intersecting each other.

[*Johnson's Folio Dic.*

The term inclination is used synonymously with angle; thus the angle or opening of two lines is called their inclination to one another.—*Penny Cyclopædia.*

An angle, in *popular language* is a corner; the point where two lines meet. In *geometry* it is usually the space comprised between two straight lines that meet in a point, or between two straight converging lines which, if extended, would meet, or the quantity by which two straight lines departing from a point diverge from each other. These are rectilinear angles.

But there are also curvilinear angles, formed by the crossing of curved lines, and mixed angles made by the meeting of one right angle with a curved line.

A right angle is an angle made by a line falling perpendicularly on another line.

An acute angle is less than a right angle, and is so called because the angular point is sharp.

An obtuse angle is an angle which has its angular point blunt or broad. It is more than a right angle.

Solid angles are formed by the meeting of more than two planes; such for example as the corner of a cube. They are such as form the boundaries of solids.

The measurement of an angle is made by considering it as the constituent fractional part of a circle. Thus we divide the circle into 360 parts called degrees. What we call a right angle is 90 degrees, or the measure of one quarter of a circle. This is also the corner of a square or right-angled paralellogram; dividing the square diagonally gives us half of ninety, or an angle of 45 degrees, and so on proportionally.

For the more accurate measurements, including those of the celestial bodies and their motions, the degree is divided into 60 parts called minutes, and the minutes are again subdivided into 60 parts, called seconds. $60 \times 60 = 3600$, there are 3600 seconds in each degree, and 360 degrees in the circumference. 3600×360 gives 1,296,000 seconds. When we speak of a second, we mean the $\frac{1}{1296000}$ part of a circumference.

Our circumference is in practice divided into 360 degrees, and all angular quantities are measured by determining them in degrees, or fractional portions of a degree, all of which are determinate parts of the circumference of a circle. In the triangulations of the United States Coast Survey hundredths of seconds are used, which are 129,600,000th parts of a circumference.

With the present excellence of our instruments and power of comparing our measurements of celestial distances with accurate measurements of time, the science of astronomy is

justly regarded as the most remarkable result of the human intellect.

Plate No. 8 is available for other illustrations. We have considered light as a radiating phenomenon, as a radiating force. Heat also radiates by a similar or analogous law. Magnetism, Electricity, Sound, Gravitation, have similar or analogous relations to the laws of measurement. All radiating phenomena illustrate and tend to explain each other.

In illustrating the radiation of heat, let the smallest circle in Plate 8 be a stove, or the half of this circle be a grate, of a cold day. If four persons at one distance (represented by the one inch,) each occupy in the curve of the fire side circle 45 degrees, the same persons at two inches distance would occupy only half of this portion; there would be room for eight at twice the distance, for sixteen at four times the distance, and for thirty-two at eight times the distance. The proportion of radiant heat will of course be diminished to the same extent in which the persons of the front rank eclipse or intercept the radiant heat. It is also obvious that this is not a simple linear relation. The persons of the front rank will intercept twice as much *breadth* of radiation as those of the second, four times as much as those of the third, and eight times as much as those of the fourth, so that here is a consideration, a measurement of both length and breadth. It belongs to the second class of measurements. Light and heat radiating in all directions, are diffused as from a centre, on all sides, and belong to the third class of measurements. But light and heat received upon a plane surface is a measurement of the second class. It is as the length and breadth of the given surface. "It is as the squares of the distances." One person in the front rank would thus intercept the radiant heat from 64 persons occupying similar space at eight times the distance.

PLATE No. 9.

Plate No. 9, although at first sight very different from No. 8, is in principle the same.

In No. 8 the squares constitute an essential part of the figure. It is on the sides of the squares that the circles are drawn. No. 9 is not different in principle or in construction, only that the squares are omitted.

In drawing the original of No. 9, the squares were first carefully drawn in pencil. The circles were then drawn, and the squares erased. The circles are therefore tobe relied on, as exactly the same as those of No. 8.

Starting from the central, or smallest circle of one inch radius, the circles double each other in area as we proceed outward toward the outer or larger circle, as is shown in the Plate 8. An ellipse is drawn, enclosing, and almost touching the smallest circle, and cutting each of the others. In pursuance of our purpose to illustrate radiant forces, let us first suppose this ellipse to represent the walls of an oval room, and let the first illustration be of light.

Let a common lamp be placed at the centre of the circles, which is one of the foci of the ellipse. The light radiating from this point would first meet the ellipse at the distance of (at a radius of) one inch, and its diffusion would be represented by the area of the smallest circle. The light reaching the ellipse at the intersection of the third, of the fifth, and of the seventh circles, is at radii of two, of four, and of eight inches respectively, and its diffusion is as the areas of these circles, that is, as the length multiplied by the breadth. The radius of the first circle is $1 \times 1 = 1$. That of the second, $\sqrt{2} \times \sqrt{2} = 2$. That of the third, $2 \times 2 = 4$. Of the fourth, $\sqrt{8} \times \sqrt{8} = 8$. Of the fifth, $4 \times 4 = 16$. Of the sixth, $\sqrt{32} \times \sqrt{32} = 32$, and of the seventh $8 \times 8 = 64$, the area of the largest circle.

The old expression is, light diffuses itself as the squares of the distances. But it must be borne in mind that as light diffuses itself, its intensity diminishes. The intensity of light is inversely as the squares of the distance. How is this? Let us consider. Light does not diffuse itself in squares. Light radiates from a centre, and its diffusion may be very naturally represented by circles. It is true, that these circles are to each other as the squares which might be drawn on their radii. It would be equally just to say that light diffuses as the octagons of the distance, or as the triangles or hexagons of the distance. It would be more intelligible, perhaps, than either, to say that light varies in intensity, diminishing as the area of diffusion increases, that is, as the length and breadth of the radii multiplied into each other. Its diffusion is obviously not a mere line. It has length and breadth. It belongs, at least, to the second class of measurements.

Light is a radiant force. Heat also is a radiant force. The same law applies to all radiant forces. Magnetism, Electricity, Sound, Gravity are all radiant phenomena.

Sound acts—reverberates—from its source or centre, in a manner which is capable of the same measurement. In approaching the ear of our audience, we gain in an analogous manner the advantage of the squares of the distance.

We may also well exemplify the mechanical construction of the wheel and the lever by this diagram. Let the ellipse be an eccentric wheel, for instance, a cam in a steam engine, its axis being the centre of the circles. A wheel is an aggregate of levers. The shortest radius of our eccentric wheel is a lever of one inch. The longest radius is a lever of nine inches. Between these we have levers of all intermediate lengths.

The eccentric wheel is thus an aggregate of levers. It may be practically considered as a short lever and a long

lever gracefully curved, so as to pass most easily from one into the other.

This diagram also illustrates perspective. If an object occupy in the field of vision at the distance represented by the largest circle, say the breadth of the portion which is colored, or $5\frac{5}{8}$ degrees, it will at the distances of the nearer circles, at the 6th, 5th, 4th, 3d, and 2d circles, occupy proportional breadths of $11\frac{1}{4}$, $22\frac{1}{2}$, 45, and 90 degrees, represented by the differently colored portions of these circles respectively.

But perhaps the most interesting application of Plate No. 9 is the exemplification of Gravity. Light and heat are examples of radiant forces whose action is centrifugal. Gravity is an example of a radiant force, whose action is centripetal. Light and heat are diffused from a centre by a law which we have exemplified. Gravity is attractive towards a centre. But the relations of both, and of all to the laws of measurement, beautifully illustrate and explain each other.

We will here transcribe or repeat our concluding remarks on the explanation of Plate No. 1. "The most important primary lesson which can be taught to the student of mathematics is the natural and direct connection of the three classes of measurements, with all the varieties and all the relations of motion.

Every motion is a result of force. The nature of force is not easily understood, but we can observe and measure the motion which is its result. One force produces motion in a straight line, and only in a straight line. Every motion which is in a straight line is a result of one single force, (or it may be of one aggregate force.) It represents, and is capable of being measured by the first class of measurements.

A motion which is not in a straight line must always be a result of more than one force. Thus, a stone thrown into the air returns to the earth. It has a curved linear motion. The

projectile force would have impelled it in a straight line, but gravity, a second force, acting upon it perpendicularly, brings it to the ground. In like manner, the movement of a stone attached to a string held in the hand, describes a curved linear motion. So also do the Planets in their revolutions round the sun, and of their satellites round the Planets."

But to return to Plate No. 9, in illustration of Gravity. If a cannon ball were fired, or a stone thrown directly upward toward the zenith, it would lose its velocity inversely, as "the squares of the distances;" (that is, as the circles of diffusion represented in the Plate, through which it was impelled,) till it reached the highest or culminating point; it would then begin to fall, and in common with all falling bodies, would acquire velocity as "the squares of the distances" through which it fell; that is, as represented by the same circles in the Plate, which we will now call circles of concentration, falling 16 feet in the first second of time, 48 feet in the second, 80 feet in the third second, and so on, and making its descent in the precise time in which its ascent was impelled. If the stone or cannon ball were thrown obliquely upwards, it would describe a curved line, but the same law of gravity would determine its ascent and descent.

If a cannon ball impelled by the greatest practicable force were fired horizontally from an elevated position, say from the fortress of Gibraltar, and at the same instant another ball were quietly dropped from the same elevation, they would fall into the sea at the same precise instant of time. The horizontal force would have propelled the first several miles, but it would have fallen as fast as the other, the action of gravity would have been exactly the same on both. The two forces in the one horizontally impelled, would not have interfered in the slightest degree with each other.

For a simpler illustration. Let us suppose a ball lying on a

table to be pushed horizontally, first by a given force in one direction, and then, after it was at rest, a second force to act upon it horizontally in another direction, say at right-angles with the first, its place of rest would be the same as if the two forces acting upon it were combined at the same instant to impel it. It would have described a path representing the diagonal of the two forces, and thus have arrived at the same point.

If we suppose one force to represent length, and the other breadth, the analogy is perfect in every possible variety and number of instances, the combined force will, both in its intensity and direction, be exactly represented by the diagonal, the hypothenuse.

Further to illustrate what are called centrifugal and centripetal forces, let us turn to Plate No. 5, and let us suppose a ball at the centre of any one of the four sides of the outer or circumscribed square. If a force act on the ball parallel with the side of the square sufficient to impel it six inches, it would rest at the corner of the square. If a second force now act on the ball at right-angles with the first, and in the direction of the line of the next side of the square, sufficient to impel it six inches it would reach and remain at the centre of the next side. But if these two forces acted on the ball at the same precise instant, it would take—its path would describe—the diagonal line (represented in the Plate by the side of the inscribed square,) to the same point. If we suppose one of the forces to be an impulse like that of the cannon ball fired horizontally from a mountain fortress, and the other a constant force like gravity, acting slowly at first, but accumulating every instant on the rapidly propelled but surely falling body, the path described would be, not the diagonal, but the curved line of the quarter circle, as in the Plate. This supposes the two forces to be equal; if they were not equal, the path could

not be a circle. It would then be some other curve, representing a portion of an ellipse, or a portion of some other section of a cone.

It obviously requires more than one force, that is, it requires two forces to sustain any body revolving around another. Gravity is centripetal, and would rapidly attract it to the centre, the equilibrium of revolution; the centrifugal force, on the contrary, if it were the only one, would carry it off in a straight line, as is usually said, in a tangent.

For a body thus to maintain a circular orbit round another, requires two forces, and these must exactly and constantly balance each other, describing an uniform curve, returning into itself, passing through uniform linear distances, and sweeping uniform and similar areas in uniform times, and with uniform angular velocity.

To sustain an elliptical orbit of revolution, the two forces need not be, cannot be always uniform and equal, as in the circular orbit, but even in this case, though the two forces are not constant and uniform as in the circle, each of the half ellipses measured from the points nearest to, and farthest from the centre of revolution, must balance and compensate each other, must (as we shall learn,) in every case be equal to, and represent each other both in linear distance, in area, and in time.

Let us now suppose the centre of the circles in Plate No. 9 to represent the sun or centre of attraction, and the circles drawn on the Plate to represent the orbits of the Planets. As the attraction of gravitation acting upon the nearer, compared with the more distant Planets, is found to be measured, not by their linear distance from the sun, but to be measured by the length multiplied by the breadth of this distance, and is represented by the areas of their respective orbits of revolution. By the same law, a body, whether a Planet or a Comet, per-

forming a revolution which is a very little or very much elliptical, being acted upon at the nearer (the perihelion) portion of its orbit, by a stronger force of gravity, and at the further (the aphelion) portion of its path by a diminished force of gravity, the centrifugal or projectile force must represent, balance, and correspondingly increase and diminish in precise analogy to that of the gravity, at every portion of its path, or the gravity which might be measured by supposing an unlimited number of circles intersecting every point of its orbit.

The orbits of the celestial bodies, whether Planets or Comets, are obviously not straight lines. They are not results of one force—they do not belong to the first class of measurements. They are the result of two forces, the centrifugal and the centripetal; they are represented by planes; they have length and breadth; they are clearly belonging to the second class of measurements.

For the just conception of the laws of their motions, we are indebted to the labors of the immortal Kepler.

We will here devote a paragraph to the history of Astronomy, previous to the time of Kepler.

Although a high antiquity is claimed for the origin of astronomical science, yet it appears certain that its earliest development during many centuries was slow, and its results limited. The ancients considered the planets and the firmament to revolve in perfect circles, the centre of which they long supposed to be occupied by the earth, and not by the sun.

The true doctrine, that the sun is the centre, was first taught by Pythagoras, in the sixth century before Christ, or more than twenty-three centuries ago. But as it was communicated privately to his disciples, and in the absence of the modern facilities of printing and intercourse, it is probable that it was not very widely adopted. Seven centuries afterward, the doctrines of Ptolemy again made the earth the

centre of the system, which error prevailed during fourteen centuries, till the true theory was revived by Copernicus, three centuries ago, in 1543, by the publication of his work, "De Revolutionibus Orbium Celestium," wherein he considers the sun to be the centre of the solar system, as taught by Pythagoras.

Copernicus appears to have believed in the perfectly circular form of the celestial orbits, as taught originally by Pythagoras, and his successors of the peripatetic school.

The half century succeeding Copernicus gave birth to the three greatest of modern astronomers. Tycho Brahe, in 1546, Galileo, in 1564, and Kepler, in 1571.

Tycho Brahe, born in 1546, at Knudsthorp, and educated at the University of Copenhagen, having witnessed when only thirteen years of age the occurrence of a great eclipse of the sun, and having seen the sun darkened at the very moment, and to the very extent which had been predicted and delineated, directed his attention to astronomy, and with great zeal, and under all discouragements, continued to devote himself to his favorite science.

When Tycho was about thirty years of age, his sovereign, Frederic II, of Denmark, a prince distinguished for liberal and scientific tastes, gave him a grant of a convenient site, the island of Huen, and erected an observatory and accommodations on a liberal scale, at an expense of one hundred thousand dollars, and Tycho is said to have expended on them a similar sum from his private fortune. The observatory and instruments erected under Tycho's own superintendence, were on a scale which so far surpassed in size, in ingenuity of construction, and efficiency, any which had previously been used, that his observations approached within a few seconds of absolute accuracy, instead of a few minutes, as had hitherto been practicable.

Here for more than twenty years, Tycho devoted himself to observations of the celestial motions with great zeal and success.

It may be remarked, that the three great epochs of astronomy, as a science of pure observation are

First, that of Hipparchus, who measured the earth from the heavens, and who has been called the founder of Astronomy, and of correct geography, 160 years before Christ, whose observations approached to correctness within one half degree.

Second, that of Tycho Brahe, A. D. 1580 to 1600, correct within two minutes, or $\frac{1}{30}$ of a degree.

The first use of the telescope, Galileo, January, 1610.

Third epoch of observation is that of Bradley, author of the discoveries of Nutation, Aberration, and of improvement in the tables of refraction, 1742 to 1762.

Bradley's Observations, observing with the long telescope, of 212 feet focal length, introduced by Huygens, are correct within two seconds, or $\frac{1}{1800}$ of a degree.

Sir David Brewster observes: "as a practical astronomer, Tycho has not been surpassed by any observer of ancient or modern times. The splendor and number of his instruments, the ingenuity which he exhibited in inventing new ones, and in improving and adding to those which were formerly known, and his skill and assiduity as an observer, have given a character to his labors which will be appreciated to the latest posterity."

The death of Frederic in 1588, and the accession of a new dynasty, after various ungracious incidents, at length sent Tycho, in 1597, an involuntary exile from his ungrateful country, and by the invitation of Rudolph, emperor of Bohemia, he visited Prague, where he was kindly received, his immediate wants provided for, and the castle of Benach, a few miles distant, was allotted for his home, with a pension of three thousand crowns.

On this occasion, an exile from his country, and the spot where he had devoted so much labor, he comforted himself with the idea that every soil was the country of a great man, and that wherever he went the blue sky would still be over his head; and dying in exile, during the temporary delirium which accompanied his last illness, frequently exclaimed, "Ne frustra vixisse videor." It is certain I have not lived in vain.

In January, 1600, Kepler paid a visit to Tycho, who received him in the kindest manner, and gave him an introduction to the emperor. After enjoying Tycho's society three or four months, Kepler returned home to Gratz. But in September, 1601, he returned to Benach, in consequence of Tycho having obtained for him the appointment of imperial mathematician, on the condition of Kepler's assisting him in his observations, and on Tycho's death, which happened in October of the same year, Kepler succeeded him as principal mathematician, and to the inheritance of his valued observations. It was thus that fortuitous or providential events, the persecutions of the age which exiled Tycho, and which subjected Kepler to continued poverty, and the proffered hospitality of Rudolph, threw these two great men into association.

Although it is certain that for a number of years previous, Kepler had had under consideration various hypotheses concerning the number, distance, and periodic times of the planets, it is equally certain that this connection was peculiarly valuable to Kepler, as the observations of Tycho were the only ones in the world which could enable him to carry on his own theoretical inquiries, and it may be that to this circumstance is mainly owing his grand discoveries.

The system which Tycho had been taught, and which he advocated, was identical with the one which Copernicus had rejected, and consisted in supposing the sun to revolve about the earth, carrying with it all the other planets revolving about

him, and indeed Tycho not only denied the revolution of the earth about the sun, but likewise its diurnal revolution on its axis.

Galileo, also, when a very young man, at the time when the Copernican system began to be discussed, and when Wurteisen, a disciple of Copernicus, delivered on this subject two or three lectures, was unwilling to be present, being firmly persuaded that this opinion was a piece of solemn folly. And it is stated by Gerhard Voss, that subsequently, a public lecture of Maestlin, the instructor of Kepler, was the means of making Galileo acquainted with the true system of the universe.

Lord Bacon, also, who was nearly a cotemporary with Galileo, positively condemned the idea of the earth's motion as absurd and false, and, it is believed, never altered this opinion.

But Kepler, who was a few years younger than these, at the college of Tubingen, under the tuition of Maestlin, had become acquainted with the true system of the universe, and delighted in the simplicity of the Copernican system.

Such is a brief outline of the theory of the universe before the time of Kepler.

The theory adopted by Kepler in the early part of his discussion of Tycho's observations, appears to have been that the orbit of each planet, including the earth's was circular, and that it was described with an uniform angular velocity, like the hands of a clock, and that not only its angular but also its linear motion in its orbit was uniform.

In proceeding to discuss the orbit of Mars. The observations of Tycho had determined the motions of this planet to a great degree of accuracy, and Kepler soon found that the inequalities of motion differed widely from any thing which could be represented by the system of epicycles, as previously suggested by Copernicus. He accordingly, with incredible diligence, directed his observations to investigate the real form of the orbit of Mars.

This was difficult, because the observations were made from the earth, itself in motion. Hence it was necessary to distinguish that *apparent* part of the inequality of the motion of Mars, which is caused by the inequality of the earth's own motion in its orbit.

After four years of laborious calculation, the non-accordance of his adopted theory of a circular orbit with his own observations, and those so carefully recorded by Tycho, had determined that the motions of Mars were not circular, that they were not alike at all portions of its orbit, that it moved at some times more rapidly than at other times, and he was, therefore, obliged to seek some other explanation.

Considering, therefore, the earth's orbit as circular, "Kepler, after many fruitless speculations, in endeavoring to explain these planetary inequalities, was led to suppose the orbit of the planet to be oval, and from his knowledge of the conic sections, he afterwards determined it to be an ellipse, with the sun placed in one of its foci. He then ascertained the dimensions of the orbit, and by a comparison of the times employed by the planet to complete a whole revolution, or any part of one, he discovered that the time in which Mars describes any arches of his elliptic orbit were always to one another as the areas contained by lines drawn from the focus, or the centre of the sun, to the extremities of the respective arches, or in other words, that the radius vector, or the line joining the Sun and Mars described equal areas in equal times. By examining the inequalities of the other planets, he found that they all moved in elliptic orbits, and that the radius vector of each described areas proportional to the times. These two great results are known by the name of the first and second laws of Kepler. The third law, or that which relates to the connection between the periodic times and the distances of the planets, was not discovered till a later period of his life."

[*Sir David Brewster.*

What is "the radius vector?" Let us be certain. The radius is a straight line from the centre to the circumference of a circle; vector (from veho) means carried or moved; the radius vector is the radius moving, the radius revolving, the revolving radius. Now the radius of the first circle, one inch in length, sweeps in one revolution the complete area of this circle. The rule is universal. The radius of any circle sweeps in one revolution its complete area. The area of the seventh circle is sixty-four times that of the first. It is obvious, therefore, that a planet revolving in an orbit represented by the first circle, to describe "equal areas in equal times," must revolve exactly sixty-four times, while another planet, revolving in an orbit represented by the seventh circle, revolves once.

We have already demonstrated that the intensity of light and of heat at distances represented by the radii of the respective circles, are as the length multiplied into the breadth of those radii. We have assumed, and it is an important truth we maintain without fear of contradiction, that all radiant phenomena illustrate each other. Centripetal phenomena inversely represent centrifugal.

We have shown (page 88,) that the centrifugal force or motion is equal to, and is compensated, balanced, indicated, measured by an equal centripetal force or gravity, at any distance respectively.

If this be true, the spaces swept over by each one of any number of bodies, revolving by the law of gravity, at various distances, whether these bodies revolve in circles or in ellipses, will represent the times respectively occupied by each. The spaces in truth represent times. The spaces are simply indexes to the times. The hand moving over the dial of a clock is a radius, it is a radius vector, and it sweeps equal areas in equal times.

Let the centre of the circle in Plate No. 9, represent the Sun. Let the first circle represent the orbit of Mercury, the second that of Venus, the third that of the Earth, the fourth that of Mars, the fifth Jupiter's, the sixth Saturn's, the seventh Herschell's. It is obvious that to sweep equal areas, the first must revolve twice while the second revolves once; four times while the third revolves; eight times while the fourth; sixteen times while the fifth; thirty-two times while the sixth, and sixty-four times while the seventh revolves once.

It is important at this point to compare Plate No. 9 with Plate No. 3, and to recognise the analogy, or rather the demonstration of the same principle, in both. In No. 3, Fig. 12, we have an aggregate of triangles, which, though different in form are demonstrably equal in area—the same quantity extended into a different form; and we remark that No. 3 gives perhaps the simplest primary idea of the laws of motion. In No. 9, the radii as colored in the plate, the radii vectores, which represent the moving radii of the orbits of the planets, at their respective distances, are no other than triangles of equal area, and in this respect of being in equal area to each other, are perfectly analogous to those in No. 3.

Plate No. 9 may also be made a means of explaining the principles of the laws of Gravity, as exemplified in the movement of that most useful and interesting invention, the pendulum.

The motion of the pendulum is in a plane. It is like the planetary revolutions, the result of two forces. One of these forces is gravity, (centripetal,) which inclines the pendulum directly to the earth's centre, the other is a lateral (centrifugal) force, produced mechanically by a spring or weight, &c. The planetary motions are celestial mechanics. The motion of the pendulum is terrestrial mechanics. But the great truth of the combined action of two forces explains both.

The planets, at different distances sweep equal areas in equal times. Pendulums of different lengths sweep equal areas in equal times. Plate No. 9 equally well exemplifies both. If we shorten the length of any pendulum in the least, or to any degree, we shall increase its lateral motion, or its speed to a compensating extent. If we shorten a pendulum to half the length, either its motion laterally, or its speed, (or the resultant of both,) will be doubled; if to one quarter the length, quadrupled. If we lengthen it, the reverse. Thus, if we have a pendulum of eight inches in length, that is, whose vibrations are represented in the Plate by the section or colored portion of the largest circle of eight inches radius, and if we shorten the radius, (the length) one half (to four inches,) or three quarters (to two inches radius,) we shall correspondingly increase the breadth. See Plate No. 9.

The author of the pendulum was Galileo, and though it will be a short digression from the continuity so important to our subject, some historical notice of its discovery may be a useful aid to the student in the comprehension of the principle demonstrated in this diagram. We will quote from Prof. Powell:

"A few years earlier than Kepler, (Feb. 15, 1564,) was born his illustrious friend and cotemporary in public fame, Galileo de Galilei, the son of Vicenzo de Galilei, of Pisa, in Italy. After the fashion of that age and country, he appears to have been more commonly designated by the name of Galileo, by which he has been since known.

"Vicenzo was a man of considerable learning and talent, who early appreciated and properly cultivated the expanding genius of his son. At the university of Pisa he was very soon distinguished, not merely for general proficiency in his studies, but for the singular boldness with which he maintained original opinions, often at variance with the received scholastic dogmas.

"One grand tenet of the schools was that heavy bodies fall to the earth more rapidly than lighter, in proportion to their weight. Galileo, in the presence of the university, ascended the leaning tower of Pisa, and dropped from its summit bodies of different weights; with an inconsiderable difference, due to the resistance of the air, they reached the ground nearly at the same moment. The learned quoted Aristotle in preference to their senses, and contended that a mass of ten pounds must and does fall in one tenth of the time occupied by the falling of one pound, and as Galileo had made them fall in the same time, the only result was an inveterate hostility against him.

"In the year 1592, Galileo published a treatise "Della Scienza Mecanica," in which, after examining the theory of the mechanical powers, he lays down the general proposition, that all the advantage given by them is simply this: they make a small force equivalent to a great one, by causing it, the former, to move over a proportionally greater space in the same time. And in following out this principle, he further shows, that if the effect of a force be estimated by the weight it can raise to a given height, in a given time, this effect can never be increased by any mechanical contrivance whatever. He proceeded also to the theory of the oblique lever and the inclined plane. This was the discovery of the well known truth, that what is gained in time is lost in power, and vice versa.

"In the cathedral of Pisa, one of the chandeliers hanging from the lofty roof had accidentally been set swinging: this instantly struck the observant mind of the young philosopher, who noticed the fact that its vibrations were performed in exactly equal times, by comparing them with his pulse. This subject afterwards occupied much of his attention, but for the present he went no further than to apply the principle to the purposes of the medical profession, to which his views were

directed, by contriving a pendulum with a variable length of string, by which its beats might be made to accord with those of the pulse, and thus give a measure of its rate.

"In the theory of motion he had, as we have seen, investigated some of the laws of falling bodies. He extended this inquiry, by first assuming that they receive equal increments of velocity in equal times, and thence deduced mathematically that the *spaces described* must be as the squares of the times, and that the space fallen through in one portion of time is exactly half that which would be described in the same time, with the velocity last acquired continued uniformly. He soon saw that a body descending on an inclined plane must be in like manner accelerated; he therefore adopted this as a simple mode of putting his theory of falling bodies to the test of experiment.

"From his knowledge of the inclined plane, it easily followed that the times of falling down, all the *chords* of a circle, terminating at its lowest point, must be the same. He however fell into an error in maintaining the same of the circular arcs, and in applying this to the vibrations of the pendulum.

"Knowing the law of falling bodies, to give spaces proportional to the squares of the times, he deduced the motion of projectiles, and showed their path to be a parabola. This reasoning involved a principle which Galileo does not expressly refer to by name, but which is, in fact, one of the cases of the general principle of the composition of forces.

"The mechanical and dynamical researches of this great philosopher are unquestionably those in which we recognise the first real union of experimental and mathematical reasoning in investigating the laws of force and motion.

"These researches, although perhaps less brilliant than his astronomical discoveries, on which his popular fame may be more universally founded, are of inestimable value."

The following able summary of his character and labors is given by Prof. Playfair:

"One forms, however, a very imperfect idea of this philosopher from considering the discoveries and inventions, numerous and splendid as they are, of which he was the undisputed author. It is by following his reasonings, and pursuing the train of his thoughts, in his own elegant, though somewhat diffuse exposition of them, that we become acquainted with the fertility of his genius, with the sagacity, penetration, and comprehensiveness of his mind. The service which he rendered to real knowledge is to be estimated, not only from the truths which he discovered, but from the errors which he detected; not merely from the sound principles which he established, but from the pernicious idols which he overthrew.

"His acuteness was strongly displayed in the address with which he exposed the errors of his adversaries, and refuted their opinions by comparing one part of them with another, and proving their extreme inconsistency. Of all men who have lived in an age which was yet only emerging from ignorance and barbarism, Galileo has most entirely the tone of true philosophy, and is most free from any contamination of the times in taste, sentiment, and opinion.

"By the writings of Copernicus, Kepler, and Galileo, the solar system, and the subordinate series of truths referring to the theory of motion, were so completely established, and universally demonstrated, that nothing was wanting but time to allow men's opinions to come gradually round to the truth.

"The lengthened career which Providence assigned to Galileo was filled up throughout its rugged outline with events of dramatic interest. But though it was emblazoned with achievements of transcendent magnitude, yet his noblest discoveries were the derision of his cotemporaries, and were

even denounced as crimes which merited the vengeance of heaven.

"His scientific character, and his method of investigating truth, demand our warmest admiration. The number and ingenuity of his inventions, the brilliant discoveries which he made in the heavens, and the depth and beauty of his researches respecting the laws of motion, have gained him the admiration of every succeeding age."

In exemplification of the Third Law of Kepler, let us once more refer to Plate No. 9.

This third law, as he himself informs us, first entered his mind on the 8th of March, 1618, but having made an erroneous calculation, he was obliged to reject it. He resumed the subject on the 15th of May, and having discovered his former error, he recognised with transport the absolute truth of a principle, which, for seventeen years had been the object of his incessant labors. The delight which this grand discovery gave him had no bounds. "Nothing holds me," says he: "I will indulge in my sacred fury. * * * "If you forgive me, I rejoice; if you are angry, I can bear it. The die is cast; the book is written, to be read either now, or by posterity, I care not which. It may well wait a century for a reader, as God has waited six thousand years for an observer."

What is the third Law? Let us recapitulate. The first Law of Kepler is, that the orbits of the Planets are not circular. That they are ellipses, of which the sun occupies one of the foci.

The Second Law of Kepler is, that the radius vector of a planet, and also that the radii vectores of a number of planets at various distances, describe equal areas in equal times. This law we have exemplified on pages 101 and 102, viz: That the orbits of the planets are planes. That they have length

and breadth only. That they can belong only to the second class of measurements.

THE THIRD LAW OF KEPLER is, that the squares of the periodic times are to one another as the cubes of the distances from the sun. The cubes of the distances? This indicates Length, Breadth, and Thickness. This is obviously the third class of measurements. How is this?

The orbits of the planets are planes. That is certain. But Gravity, acting from the sun is not a mere plane. It is acting on all sides spherically. It is diffusing itself in spheres. Its intensity, like that of light and heat diminishes as the length, breadth, and thickness of its radii of diffusion, as the cubes of the distances. True! true! that's it. It belongs to the third class of measurements.

It might be expressed thus: The plane circles or ellipses in which the planets are revolving, are to each other as the corresponding spheres in which gravity exerts or diffuses itself.

Or, the squares of the numbers which express the times of revolution, are to one another as the cubes of those numbers which express the mean distances of the planets from the sun.

Or, the length and breadth (the squares) of the plane orbits (circular or elliptical) in which the planets are revolving, are to each other as the length, breadth, and thickness of the spheres in which gravity exerts its intensity.

Let us observe Plate No. 9. Let the centre of the circles represent the sun, and the various circles represent orbits of the planets. Let each of the circles represent a sphere in which gravity is exerting itself, and around which the orbit of the planet is described. Circles are to each other as the length multiplied by the breadth of their radii; that is, as the squares of their radii. Spheres are to each other as the length multiplied into the breadth, and again this product

multiplied by the thickness; that is, as the cubes of their radii, the cubes of the distances. In this view, Plate No. 9 is a mere section of a nest of hollow spheres fitting into each other.

Sir John Herschell thus treats of this subject, Chap. VIII, page 262:

"In casting our eyes down the list of the planetary distances, and comparing them with the periodic times, we cannot but be struck with a certain correspondence. The greater the distance, or the larger the orbit, evidently the longer the period. The order of the planets, beginning from the sun, is the same, whether we arrange them according to their distances, or to the time they occupy in completing their revolutions, and is as follows: Mercury, Venus, Earth, Mars, the four zodiacal planets; Jupiter, Saturn, and Uranus. Thus, the period of Mercury is about 88 days, and that of the Earth 365, being in proportion as 1 to 4 .15, while their distances are in the less proportion of 1 to 2 .56, and a similar remark holds good in every instance. Still the ratio of increase of the times is not that of the squares of the distances; it is not so rapid.

"The square of 2 .56 is 6 .5536, which is considerably greater than 4 .15. An intermediate rate of increase between the simple proportion of the distances, and that of their squares, is thus clearly pointed out by the sequence of the numbers, but it required no ordinary penetration in the illustrious Kepler, backed by uncommon perseverance and industry, at a period when the data themselves were involved in obscurity, and when the processes of trigonometry and of numerical calculation were encumbered with difficulties, of which the more recent invention of logarithmic tables has happily left us no conception, to perceive and demonstrate the real law of their connection.

This connection is expressed in the following proposition:

"The squares of the periodic times of any two planets are to each other in the same proportion as the cubes of their mean distances from the sun." Take for example, the Earth and Mars, whose periods are in the proportion of 365 .2564 to 686 .9796, and whose distances from the sun are in the proportion of 100 .000 to 152 .369, and it will be found by any one who will take the trouble to go through with the calculation, that $(365\ .2564)^2 : (686\ .9796)^2 :: (100\ .000)^3$ to $(152\ .369)^3$"

[The expression of this law of Kepler requires a slight modification, when we come to the extreme nicety of numerical calculation, for the great planets, due to the influence of their masses. This correction is imperceptible for the Earth and Mars.]

Sir John Herschell proceeds:

"Of all the laws to which induction from pure observation has ever conducted man, this third law (as it is called) of Kepler may justly be regarded as the most remarkable, and the most pregnant with important consequences. When we contemplate the constituents of the planetary system from the point of view which this relation affords us, it is no longer mere analogy which strikes us, no longer a general resemblance among them, as individuals independent of each other, and circulating about the sun, each according to its own peculiar nature, and connected with it by its own peculiar tie. The resemblance is now perceived to be a true *family* likeness; they are bound up in one chain, interwoven in one web of mutual relation and harmonious agreement, subjected to one pervading influence, which extends from the centre to the farthest limits of that great system, of which all of them, the earth included, must henceforth be regarded as members."

PLATE No. 10.

Plate No. 10 does not differ in principle from No. 9; the circles are the same, and the colored portions are similar in area, and only differently arranged. It is designed in aid of No. 9, further to exemplify the same truths, the laws of Kepler, and the generalization of these laws by Cassini, and by Sir Isaac Newton.

The third Law of Kepler was discovered on the 15th of May, 1618. Half a century later, these laws of Kepler received a most beautiful confirmation in their application to the motions of the satellites of Jupiter and Saturn, from the careful observations of the elder Cassini, who was invited from Italy into France by Louis XIV, in 1669, and established in the observatory at Paris, where he continued a long series of accurate and valuable labors, particularly directing his attention to perfecting the theory of the interesting system of Jupiter, which so beautifully represents, on a small scale, the greater planetary system. He also discovered, ultimately, four satellites of Saturn, in addition to the one observed by Huyghens.

Another beautiful verification was afforded, eleven years later, by the observations of Newton, on the brilliant comet of 1680, which determined that its orbit was a parabola, and that its rate of motion also accorded with this law of the equable description of areas in equal times. Comets range into our system from all quarters of the heavens, and from distances inconceivably great in the depths of space. Hence gravitation extends in all directions, and to unknown and inconceivably great distances from the centre of our world.

But it was two years later, in 1682, that Newton applied the same laws to the motion of the Moon round the Earth, and its connection with that particular physical property

which we find belonging to particles of matter on the earth's surface, called gravity, or weight, which application constitutes his most valuable and remarkable achievement.

The orbit and distance of the moon had been so well ascertained by the long continued labors of astronomers, that with respect to them there could be little doubt. Not so the magnitude of the earth. Picard's more accurate determination of the earth's measurement, founded upon arcs of the meridian, was the subject of discussion at a meeting of the Royal Society, in June, 1682.

The raging of the plague in 1665-6, induced Newton to quit Cambridge, and retire to Woolsthorpe in Lincolnshire, his native parish. It was here that he began to reflect upon the nature of the force by which bodies at the earth's surface are drawn towards its centre, and to conjecture that the same force might possibly extend to the moon, and there be of sufficient intensity to counteract the centrifugal force of that satellite, and thereby retain it in its orbit about the earth. To compare this hypothesis with observation, it was necessary to determine the law according to which the intensity of such a force would vary with the distance from the earth's centre. In other words, simply to apply the law of Kepler, and thus generalise its application further than had yet been done. But Newton, at that time, in 1666, failed to establish the truth of this theory of gravitation, in consequence of his employing an erroneous measure of the earth's radius—a degree of latitude being then estimated at 60 miles instead of 69½, its more correct length. Years passed on. It was about 1679, when Picard, one of the most ingenious and faithful of observers, (originally a gardener) instructed by the astronomers Le Valois and Gassendi, and who furnished the plan of the observatory of Paris, made the first application of the telescope to the measurement of terrestrial angles. The superiority

of his instruments and methods, and the care with which his operations were conducted, inspired deserved confidence in his results. Picard, while his friend and cotemporary Cassini was applying his telescope to the determination of the orbits and motions of the satellites of those distant planets, Jupiter and Saturn, was directing his attention to the earth's measurement, and at length made a successful determination of an arc of the meridian.

It was three years after this, that, as we have mentioned, the subject was discussed at a meeting of the Royal Society, in June, 1682, and Newton having taken a note of the result, and thus deduced the length of the earth's radius, eagerly resumed his former calculation, and having now the necessary element for its verification, he revived his former hypothesis, which had lain dormant for sixteen years.

These laws of Kepler were discovered before the invention of the telescope. The progress of this discovery, although Galileo had been condemned to perpetual imprisonment, and Kepler died in poverty and starvation, was onward; the observation of the truth of these laws could not fail to excite the interest of astronomers. They were found to be true, absolutely true, but at the same time subject to some very remarkable temporary disturbances.

It was found that when two planets approached nearest to each other, that is, when they were both on the same side of the sun, that they exerted upon each other a reciprocal attraction. Their approach towards each other, as the inner planets (or planets nearer to the sun) revolved more rapidly, was always by the nearer planet overtaking, as it were, the outward planet in its less rapid revolution, and it was found that the nearer or overtaking planet was accelerated in its orbit, while at the same time the overtaken planet was retarded. This effect was observable until the two planets

were in conjunction, that is, until both planets were on the same side of the sun, and at the point of their orbits opposite and nearest to each other, or until (if they were revolving in the same plane, which they are not,) a line might be drawn from the sun through the centre of both planets. Now the reverse effect occurred; the mutual attraction still existed, but it was changed; the outer planet, whose motion had hitherto been retarded by the gravity of the nearer or overtaking planet was immediately accelerated, and the inner or overtaking planet was no longer accelerated, but to a corresponding extent retarded. And it was found that the mutual acceleration and retardation exactly balanced, and were reciprocally compensatory to each other, and that thus both planets were restored to their own true orbit and motion of equal areas in equal times, as though no disturbance had happened.

In two planets (by supposition) of similar weight, the acceleration and retardation would be both compensatory and equal, one would be accelerated as much as the other would be retarded. But if either planet, say the accelerated planet, were in any proportion heavier than the other, say ten times heavier, it would retard the overtaken planet ten times as much as it would itself be accelerated. But the instant it passed the point of conjunction it would commence, in the same ratio the compensating process; it would now accelerate the lighter one ten times as much, restoring it to its independent rate of motion, while its own retardation would be only one tenth, or in a ratio precisely compensatory.

The comparative reciprocal attraction indicated by this disturbance in the time of each other's motion, affords a very perfect means of comparing the weight of the planets, weighing them against each other, as it were on a steelyard.

The consciousness of this truth as regards the known planets, was the means of the discovery of the last outside

planet Neptune, which, though it is a large planet, yet from its vast distance, might have otherwise remained forever undiscovered among the fixed stars.

The planet Herschel, or Uranus, was discovered by Sir William Herschell, in 1781. Though eighty-two years, or *earth's revolutions* have since passed, this planet has not to this time (1853,) completed an entire revolution round the sun; its entire revolution requiring a little more than eighty-four years. But its path has been carefully observed by Sir William Herschell, and a numerous succession of astronomers. The conviction that a disturbance of the orbit of this planet by a more distant planet was not improbable, induced observers to be in some expectation of such an event This expectation was at length confirmed by the partly simultaneous observations of a considerable number of astronomers, of whom Le Verrier was the more fortunate, in being the first to announce to the world the existence of another and more distant planet, Le Verrier or Neptune.

Le Verrier, in directing his telescope to Herschell, observing that its rate of motion was accelerated, and differed from the calculated rate as laid down by previous astronomers, at once inferred the existence of an exterior planet. This inference was confirmed by each succeeding observation, and he was soon able to approximate by calculation the direction of the disturbing planet within 54' 45" of heliocentric longitude, and 30' 54" of latitude. He wrote to M. Galle, assistant at the Berlin observatory, whose means of observation were better than his own, informing him of this discovery. Dr. Galle received this letter by the post of the 23d Sept. 1846, and setting the Berlin equatorial telescope in the direction pointed out by Le Verrier, made the actual optical discovery of the planet on the same evening.

The discovery of this new and remote planet affords a case

in which the facility of application of the known laws of equal areas in equal times is capable of being exemplified. If while the earth performs an entire revolution in its orbit, this planet is found to have swept only one hundred and sixty-fourth part of the circumference of its orbit, it gives the data (provided the orbit be assumed as circular) for determining the distance of Neptune from the sun or centre of gravitation, compared with the distances of the other planets.

The mutual gravitation of all bodies in proportion to their weight being admitted, it was evident that while the planets were describing their orbits round the greatest and most powerful body in the system, they must mutually attract one another, and thence in their revolutions some irregularities, some deviations from the description of equal areas in equal times, and from the laws of elliptic motion might be expected.

Such irregularities, however, had not been observed by the earlier astronomers in the motion of any of the planets, except the moon, where some of them were so conspicuous as to have been known to Hipparchus and Ptolemy. Newton, therefore, in consideration of this subject was very naturally led to inquire what the different forces were which, according to the laws just established, could produce irregularities in the case of the moon's motion. Beside the force of the earth, or rather of the mutual attraction of the moon and earth, the moon must be acted on by the sun, and the same force which was sufficient to bend the orbit of the earth into an ellipse, could not but have a sensible effect on the orbit of the moon. Here Newton wisely observed that it is not the whole of the force which the sun exerts on the moon that disturbs her motion round the earth, but only the difference between the force just mentioned and that which the sun exerts on the earth, for it is only the difference that affects the relative position of the two bodies. To have exact measures of the

disturbing forces, he supposed the entire force of the sun on the moon to be resolved into two, of which one always passed through the centre of the earth, and the other was always parallel to the line joining the sun and earth, consequently, to the direction of the force of the sun on the earth. The former of these forces being directed to the centre of the earth, did not prevent the moon from describing equal areas in equal times round the earth. The effect of it, however, he showed to be to diminish the gravity of the moon to the earth by about one 358th part, and to increase her mean distance in the same proportion, and her angular motion by about one 179th.

From the moon thus gravitating to the centre of the earth, not by a force that is exactly inversely as the squares of the distances, but by such a force diminished by another contravening force, it was found, from a very subtle investigation, that the dimensions of the elliptical orbit would not be sensibly changed, but that the orbit itself would be rendered moveable, its longer axis having an angular and progressive motion, by which it advanced over a certain arc during each revolution of the moon. Its motion was a revolution round the earth, itself revolving round the sun. This afforded an explanation of the motion of the *apsides*, that is, the line connecting the greatest and least distance of the moon from the earth, which had been observed to go forward, during each time of the moon's revolution at the rate of three degrees and four minutes nearly, in relation to the fixed stars.

In short, Kepler, in 1601, previous to the invention of the telescope, had achieved his great discovery of the laws by which the motions of the planets, including the earth, revolve round the sun. About 1670, Cassini, by the aid of the telescope, had applied these laws to the revolutions of the satellites of Jupiter and Saturn round those remoter planets. In 1680,

Newton had himself verified the same laws in his observations on the elliptical orbit of the great comet of that year. In 1682 he had only to apply the same law to the earth's satellite, the moon, and also to falling bodies at the earth's surface, and thus further to extend and to generalise what is called the theory of universal gravitation.

PLATE No. 11.

In Figure 1 we have the square inch, and the square drawn on its diagonal, doubling (as by the problem of Pythagoras,) the area of this square. It is, thus far, similar to Plate No. 8, the squares of which Plate are successively in the same relation, doubling each other's areas in the ratio of 1, 2, 4, 8, 16, 32, 64, &c., which is what is technically called a geometrical series.

In Plate No. 8, the length and breadth (that is the sides,) of the triangles are in every case equal to each other. It is a simple repetition. It is now our purpose to exemplify that, however the length and breadth (that is the sides,) of the triangles be varied, the principle is invariable; the diagonal (the hypothenuse) will in every case be a result of the length and breadth. It is a function of the length and breadth.

In Plate No. 11 we show that we can present a series of squares which shall have to each other the relation in area of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and so on, in an infinite series, which is technically called an arithmetical series. An arithmetical series is where the increase is always equal.

In Figure 2 we first repeat Figure 1. If we now, on the right, add another square inch at right-angles with the side of the second square, and draw the hypothenuse, the square which may be drawn on this must be equal to the other two squares. We know these to be of one and two inches area,

and it is evident that this resulting square is of three inches area. For the dissection of this square, and the re-arrangement of the three square inches on the extreme right, the author is indebted to an ingenious friend.

Above, on the side of the third square, we again add, at right-angles with it, another square inch, and draw the hypothenuse. It is evident that the square on the hypothenuse is now four inches.

Proceeding to Figure 3, we add to square No. 4, at right angles, another inch, and the square on the hypothenuse is five inches area. This square may be analysed into the one square inch in the middle, and the four similar triangles which surround it. Each of these triangles being the half of parallelograms of two inches length by one inch breadth, are (by Plate No. 2,) one square inch in area. Four triangles of one inch and one square inch in the middle = 5.

Figure 4 presents another example of the same invariable problem. To a square of three inches sides = 9 inches area, we add, at right-angles, a square inch, and draw the hypothenuse. Our triangle is now three inches length, by one inch breadth. Its area is $1\frac{1}{2}$ inches. The square drawn on the diagonal or hypothenuse, is 10 inches area. This may be analysed into the square of two inches sides, four inches area in the middle, and the four triangles or half parallelograms surrounding it. Each of these is three inches in length by one inch in breadth, and consequently contains $1\frac{1}{2}$ square inches area. $4 \times 1\frac{1}{2} = 6 + 4 = 10$.

Figure 5 presents another example. To a square of four inches sides, $4 \times 4 = 16$, we add, at right-angles, a square inch, and draw the diagonal. Our triangle is now four inches length by one in breadth. The square drawn on the hypothenuse is 17 inches area. This may be analysed into the square of three inches sides = 9 inches area in the middle,

and the four surrounding triangles or half parallelograms. Each of these is four inches in length by one inch in breadth, and consequently contains 2 square inches in area. $4 \times 2 = 8 + 9 = 17$.

Figure 6 comprises Figures 1, 3, 4, and 5, combined in one. In Figure 6 each of these may be contemplated separately. We may also demonstrate the area of each of these squares from the other, either synthetically, starting from the smaller squares, or analytically, starting from the larger. In either case, the triangles added or cut off are the precise counterparts of the same parallelogram.

If we start from the largest square of four inches sides, 16 inches area, 16 inches area, less four triangles of $1\frac{1}{2}$ inches area, = 10 inches area. Or, if we start from the middle square of four inches area, added four triangles of $1\frac{1}{2}$ inches = 10 inches area. $16—(1\frac{1}{2}\times 4)=10$. $4+(1\frac{1}{2}\times 4)=10$. If we start from the square inch in Figure 6, which corresponds to the middle square in Figure 3, it is $1+(1\times 4)=5$. $9—(1\times 4)=5$.

PLATE No. 12.

Plate No. 12 presents, combined in one connected figure, a series of examples of the same problem illustrated in Plate No. 11.

On the left, starting from the square inch, we first draw the square on its diagonal = 2 inches area. To this, at right-angles, we add another square inch, and draw on the hypothenuse the third square of three inches area, and again, in the same manner, $3+1$=the square of four inches area; the fourth, $4+1=5$; the fifth $+1=6$; the sixth $+1=7$ inches in area.

To the seventh square we add, exactly as before, at the

further side, the one square inch, and draw on the hypothenuse the square No. 8.

We now alter the plan, but not the principle of our figure. On the nearer side of square No. 8 we add the one square inch, and draw the diagonal. On this we build square No. 9, of nine inches area. On the nearer side of square No. 9, again adding one square inch, we draw the diagonal, and form square No. 10 = 10 inches area. In like manner we form successively squares 11, 12, 13, 14, 15, 16, each of them increasing in area one inch upon the preceding one, and all of them being examples of the invariable problem of Pythagoras.

It is also obvious that we might continue to add, in a similar manner, the one square inch without limit, thus forming an infinite series of squares increasing in the same ratio.

PLATE No. 13.

Plate No. 13, like Plate 12, presents another series of examples of the same problem of Pythagoras, forming one of a system of spirals which is capable of unlimited variety, and when fully understood will be found to have considerable novelty, utility, and beauty.

Drawn one inch to the right from the centre, we have the square inch. On the diagonal of this square we build the square of two inches area, the length of the side of which is expressed by the square root of two, or, in short hand, $\sqrt{2}$. On the near side of this square, we again add the square inch, and, drawing the hypothenuse, build the square No. 3, of three inches area, the side of which is the square root of three, $\sqrt{3}$. Proceeding from square No. 3, in the same manner we form successively squares No. 4, 5, 6, 7, 8, and so on, up to 16, which is all we have drawn. But it is obvious that we

might go on in a similar manner without any limit, that the progression might be extended to hundreds, thousands, or millions, in short, that it is capable of an *infinite series*.

This diagram, with the six which follow, being valued by their author as important demonstrations of geometrical truths of which Pythagoras was the first discoverer, are with gratitude inscribed to the memory of that great philosopher.

Pythagoras flourished about 550 years before Christ. He was a pupil of Thales, the earliest of Greek geometers. To him the world is indebted for the discovery of that remarkable problem which constitutes the forty-seventh proposition of the first book of Euclid, viz: that the square described upon the hypothenuse is equal to the sum of the squares described upon the other two sides; a demonstration of the highest importance, from the very numerous applications it finds in every branch of mathematical science. Prof. Powell rightly designates this the fundamental theorem of geometry, and justly remarks that the story related of its author's sacrificing a hecatomb for joy, on its discovery, would hardly have been invented if he had not at least enjoyed the reputation of originality in the discovery.

The merits of Pythagoras were fully acknowledged by antiquity. When the Romans, in the 411th year of the city were commanded by the oracle of Delphos to erect statues to the bravest and wisest of the Greeks, the distinguished honor was conferred on Alcibiades and Pythagoras.

To Pythagoras Philosophy is indebted for the name it bears. Previous to his time it had been the custom among the Greeks to call men of observant and reflective minds σοφος, *wise*; he took the more modest title of φιλος-σοφια, "a lover of wisdom."

"He had studied under Pherecydas, a disciple of Pittachus; but to neither of these does it appear that he was indebted for any new knowledge of mathematical or physical subjects.

On his return from his travels in Egypt and the East, in the time of the last Tarquin, finding his native country, Samos, under the dominion of the tyrant Polycrates, he went as a voluntary exile, to seek a tranquil retreat in Italy. At Crotona, (as we learn from Ovid,) he studied and taught those sublimer views of the material universe, into which an insight is to be acquired by the rejection of artificial systems, and a free communing with nature in her own domains. Pythagoras seems to have clearly understood what we now distinctively call the *solar* system of the world, and to have recognised the diurnal rotation as well as the annual revolution of the earth, the central position of the sun, and the revolutions of the planets, to which he added a just idea of the nature of comets. All this, however, was communicated only to his disciples in private. It has also been supposed that he, or his immediate followers taught the probable existence of other systems, of which the fixed stars were the suns.

PLATES No. 14 & 15.

Plate No. 14, and Plate No. 15, are in truth the same Plate as No. 13. The shading and color used may, perhaps, have great utility in making a lucid impression on the eye, and on the minds of young and unitiated students, but it is really not necessary to the diagram, intellectually considered, that is if persons would exert their intellect; though it may probably be an important aid in making the subject attractive and comprehensible to many for whom this treatise is designed.

The purpose of these diagrams will be better understood by the student after the consideration of several of those which immediately follow.

PLATE No. 16.

Plate No. 16 is also in outline precisely the same as No. 13. No. 14 and 15 are so colored as to designate the series of

squares. In No. 16 the squares are neglected. It is the especial purpose of this diagram to show the triangles on which the squares are drawn. The first triangle extends in length from the centre one inch to the right. Its breadth, based on the first square inch, is also one inch. It is the half of a parallelogram of one inch length and one inch breadth. Its area is the half of one square inch. Its length is the side of a square inch. Its breadth is also the side of a square inch. The side of a square inch is the same as the square root of a square inch. Its area is expressed in short hand $\frac{\sqrt{1}\times\sqrt{1}}{2}$

The second triangle has obviously for its length the diagonal of the first square inch, which is the side of the square of two inches area. Its breadth being one inch, its area may be expressed by $\frac{\sqrt{2}\times\sqrt{1}}{2}$

The third triangle has for its length the side of the square of three inches area. Its breadth being one inch, the side of one inch square. Its area is $\frac{\sqrt{3}\times\sqrt{1}}{2}$

The fourth triangle has for its length the side of the square of four inches area, or two inches length. Its breadth, like all the others is one inch. Its area, consequently, is $\frac{\sqrt{4}\times\sqrt{1}}{2}=$ one square inch. The length of this triangle being two inches, its breadth one inch, its area consequently is one inch, or twice that of the unit triangle No. 1.

The fifth triangle is $\frac{\sqrt{5}\times\sqrt{1}}{2}$ so that the series is $\frac{\sqrt{1}\times\sqrt{1}}{2}$ $\frac{\sqrt{2}\times\sqrt{1}}{2}$ $\frac{\sqrt{3}\times\sqrt{1}}{2}$ $\frac{\sqrt{4}\times\sqrt{1}}{2}$ $\frac{\sqrt{5}\times\sqrt{1}}{2}$ $\frac{\sqrt{6}\times\sqrt{1}}{2}$ $\frac{\sqrt{7}\times\sqrt{1}}{2}$ $\frac{\sqrt{8}\times\sqrt{1}}{2}$ $\frac{\sqrt{9}\times\sqrt{1}}{2}$ &c. &c.

For the length of the ninth triangle we have the side of a square of nine inches area, or three inches sides, that is, a length of three inches. As we have a length of three inches and a breadth of one inch, the area of this triangle is the one half of the parallelogram of 3×1, or $1\frac{1}{2}$ inches.

Having now arrived at the third square number, we have tripled the length of the triangle, and the breadth being a constant one its area is obviously three times that of the first or unit triangle.

The next triangles are in area $\frac{\sqrt{10} \times \sqrt{1}}{2}$ $\frac{\sqrt{11} \times \sqrt{1}}{2}$

$\frac{\sqrt{12} \times \sqrt{1}}{2}$ $\frac{\sqrt{13} \times \sqrt{1}}{2}$ $\frac{\sqrt{14} \times \sqrt{1}}{2}$ $\frac{\sqrt{15} \times \sqrt{1}}{2}$ $\frac{\sqrt{16} \times \sqrt{1}}{2}$

Having now reached the fourth square number, we have obviously quadrupled the length and area of the unit triangle.

We may thus continue an infinite series of triangles, the breadth of which would always be a constant and unvarying quantity—one inch, while the length is extending continually, and without limit, but by a rate of increase which is constantly diminishing, thus presenting a curious example of infinite series, in which also the angular measurement is diminishing in an analogous ratio.

The ratio of increase of the areas of the squares in Plates No. 13, 14, 15, and 16, is an arithmetical one. Each square is in area one inch, one unit larger than the square which precedes it. But the triangles in No. 16 are in a very different ratio. They are all of equal, constant, and unvarying breadth, and consequently the increase of their area must arise entirely from the increase of their length. The triangles increase in length as the sides (the square roots) of the squares on which they are drawn. Though each is larger than the one which precedes it, it is obviously only when each square number is completed that another unit of area is completed to the triangle.

The breadth being constant, it is obvious that as often as the side of the increasing square becomes a square number, there will be added one inch in length to the triangle, and consequently one half inch in area; but the series of square numbers is 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, and their *difference* being 3, 5, 7, 9, 11, 13, 15, 17, 19, increasing by two continually, the number of triangles intervening between each square number is constantly increasing in the same ratio. Thus, though each triangle is larger in area than the last, *the ratio of increase* is continually diminishing. For instance, the square of 100 is $100 \times 100 = 10{,}000$.

Of 101 is $101 \times 101 = 10{,}201$.

The square of 1,000 is $1{,}000 \times 1{,}000 = 1{,}000{,}000$.

Of 1,001 is $1{,}001 \times 1{,}001 = 1{,}002{,}001$.

Now, though there are 200 triangles intervening between the one of 100 inches in length, and that of 101, and 2,000 triangles intervening between the one of 1,000 inches in length, and that of 1,001, and though each of these triangles is larger than the one that precedes it, yet it is obvious that the amount of area added, in any extent to which this series might be carried, is one half inch area to every additional inch in length added to the side of the square which corresponds to and measures the triangle in the series.

It may also be observed that while the areas of these triangles is thus increasing in an infinite series, their angular measurement at the point of convergence. at the centre of the diagram, is *diminishing* in a ratio precisely analogous to their increase in area. To explain: the angle of the first triangle, the diagonal of the first square is obviously 45 degrees. The angle of the fourth triangle (the second square number, and when the area of the triangle is doubled) is one half of 45, or $22\frac{1}{2}$ degrees; the angle of the ninth triangle (the third square number, and when the area is tripled) is one third of 45, or

15 degrees; of the 16th triangle (the fourth square number, and when the area is quadrupled) is one quarter of 45, or 11¼; of the 25th triangle (the fifth square number, the area is five times,) the angle is one fifth of 45, or 9 degrees; 36th triangle, (the 6th square number,) the area is 6 times, angle one sixth of 45, or 7½ degrees, and so on, in infinite series. Still, as there is always an undiminished breadth of one inch, the angle, though it may be indefinitely diminished, can never be exhausted. If there was *no breadth* there could not be an angle. It would be length only, a line. But as there is breadth there must be an angle; an angle is but a measure of length and breadth; a diagonal is a measure of length and breadth.

PLATE No 17.

Plate No. 17 is also in outline precisely similar to the four which precede it, but in addition, circles are drawn, the side of each of the squares forming the radius of each of the circles. Thus, the first circle has for its radius the side of the first square, the square inch; the second the side of the square of two inches area, the third that of the square of three inches area, &c. The ratio of the squares to each other is known, and that the ratio of the circles to each other is the same as the squares, is also known from the previous demonstration.

We have repeatedly stated as an axiom, that length multiplied by breadth is the simple, universal, invariable law of all superficial figures, and we have also, in Plates 8, 9, & 10, made some progress in demonstrating that superficial areas represent, and constitute the measure of all rotary motion, whether in terrestrial or celestial mechanics. We say length multiplied by breadth. But "multiplication is but compendious addition," (see page 37.) *Length and Breadth*, length

added to breadth, mean the same thing, and identify in all applications to mechanics, the second class of measurements.

Plate No. 17 has the same uses as Plate No. 8. It is to show that the relation of the squares, and of the sides of the squares, the radii of the circles, and the rotary and radiant forces which these represent in all their applications to terrestrial and celestial mechanics, are capable of being compared and thus measured geometrically, arithmetically, and algebraically. It is designed to aid the thoughtful student in forming a just and full conception of the laws of forces, and the mode in which the human intellect has subjected these to measurement. To connect, as it were, in one view, the geometry of Pythagoras with the levers of Archimedes, and the astronomy of Kepler, as well as with the arithmetical basis of those algebraic formulas in which the same grand truths of Nature have been clothed by Newton, La Grange, and La Place.

PLATE No. 18.

Plate No. 18 does not differ in outline from the five preceding plates. In Plate No. 17 the sides of the squares form the radii of the circles respectively. In Plate No. 18 the sides of the squares form the diameters of the circles. The circles in Plate No. 18 are half the diameter, and consequently are in area one quarter of the circles in Plate No. 17. Plate No. 18 is merely an amplification, a variety of Plate No. 17, to which it is subordinate.

PLATE No. 19.

Plate No. 19 is an amplification of Plate No. 8 and Plate No. 17. It is designed to show that the principles illustrated, (the problem of Pythagoras, the levers of Archimides, the laws of Kepler,) are capable of unlimited application.

The outline of the last six preceding plates forms the central series of Plate No. 19. In this, the added square is the inch square, added and repeated as often as may be desired, without limit in infinite series, exactly as in Plate No. 13, and the others.

But the length and breadth of the added squares, or of any squares whatever, may be diversified indefinitely without altering the principle. Thus, in the inner or smaller series, the breadth is only half an inch. Half an inch in length, and half an inch in breadth is one quarter of a square inch in area, so that only one quarter of a square inch is added at each addition to the increasing square drawn on the hypothenuse. In the outer or larger series the breadth is $1\frac{1}{2}$ inches; $1\frac{1}{2}$ inches in length, and $1\frac{1}{2}$ inches in breadth is a square of $2\frac{1}{4}$ inches. So that in this series $2\frac{1}{4}$ square inches in area is added at each addition to the increasing square.

Wherever and whenever we have any breadth we have area. If we have no breadth, nothing but *length* represented by a mere line, we have no area, and no hypothenuse. The hypothenuse represents length and breadth. It is a function of length and breadth. The hypothenuse also represents the combination of (composition of) two forces, as the centripetal and centrifugal force in the planetary motions. The composition of two forces geometrically exhibited is but length and breadth. The exposition of radiant phenomena is represented by length, breadth and thickness. It is in this connection, its relation to terrestrial and celestial mechanics, as forming the means of subjecting terrestrial and celestial motions to arithmetical comparison, that the study of the hypothenuse, the immortal problem of Pythagoras, the fundamental problem of geometry, acquires its real importance.

It is capable of unlimited application and of infinitesimal measurement, of infinitely minute as well as of infinitely

extended applications to the phenomena of the material universe.

Having thus attempted in the best manner that seems at present practicable, to lead the reader step by step from the simplest elementary considerations of measurement, to a grand and general conception of celestial mechanics, the author will here introduce an original poem, which, although its subject is Light, and not Gravitation, yet from the analogy which exists in the law of their distribution, may be of service to the student in acquiring a just conception of the grand area in which the Author of creation has established the range of universal Light, as well as of universal Gravitation.

THE CREATION OF LIGHT.

"And God said, 'Let there be Light, and there was Light.'"—Gen. i. 3.

Light is certainly the most magnificent prototype which physical creation affords of the omnipresence of Deity, and of the unity of that self-existent universal energy, which, while it "spreads undivided," and "operates unspent" in the worlds of matter and of mind, is continually revealing to intelligent creation the infinity and benignity of creation's Author.

This sublimated ethereal substance, according to the inspired Hebrew historian, was spoken into existence by the first fiat of the Eternal, and originated in the incipient act of creative energy on rude and darkling chaos. Holding, as it were, an intermediate relation to matter and immaterial existence, and triumphing in the inconceivable velocity and infinite extent of its emanations over time and space, it may be contemplated as a sort of angelic messenger from the throne of the Omnipotent to the boundlessness of universal creation.

Huygens, a Dutch astronomer of the seventeenth century, suggested that there might be "stars at such an immense distance, that their light had not yet travelled down to us since the Creation."

This hypothesis has not been disproved; it is in fact sustained by the observations which the elder Herschel, and the more modern astronomers, by means of their gigantic telescopes, have been able to make in the remote ultra-planetary space.

The idea of Huygens pictures but a solitary pencil of light, travelling from some immensely distant star for six thousand years, at the velocity of twelve millions of miles per minute, but not yet having reached us, to make its origin visible at our Earth. It is only *generalizing* this idea to believe, that the first created light of our sun, and that of every other star in the firmament, still extends itself in all directions into illimitable space; maintaining, in all its wondrous inter-radiations, the distinct proportionate visibility of its innumerable suns, at any and every point of infinite space where the eye of an observer or the lens of a telescope can be imagined to exist.

I.

Instant, o'er heaven, from every sun and star,
  Flash'd the centrifugal ethereal Light;
The planets glittered in their paths afar,
  And darkness vanished at the dazzling sight.

II.

And thence far—far that darkness hastes away,
  Where our sun's starlight dawns in boundless space;
And many a distant sun's primeval ray
  Toward us is speeding on its pauseless race.

III.

A hundred millionth part of Solar light
Illumes our planets in their kindred race;
All, save this fraction of its radiance bright,
Glances beyond them into boundless space.

[The portion of the sun's radiant light which is intercepted by and received upon the earth, is less than one twenty-three hundred millionth of the whole. It would require more than twenty-three hundred and thirteen millions of such orbs as our earth to form a hollow sphere, the radius of which would be the earth's distance from the sun.

[The proportion of the whole sphere of radiant sunlight received upon the disks of all of the planets is less than one two hundred and thirty millionth. It follows that two hundred and twenty-nine millions nine hundred and ninety-nine thousand nine hundred and ninety-nine similar portions of the whole sunlight pass beyond the limits of the solar system, inter-radiating in illimitable space with the light of innumerable other suns, the fixed stars of the firmament.

[If "nothing is wasted," the purpose of the whole sunlight, (except that comparatively inconsiderable infinitesimal portion, the one two hundred and thirty millionth, which is caught on the surface of the planets,) must be sought in the universe beyond the solar system, where, reciprocating with the light of other suns innumerable, it may have the more intellectual utility of revealing the existence of our sun, the unity of the universe, and the omnipresence of the Creator to the (supposed) inhabitants of other systems, the central suns of which, anchored in infinite space, appear to our limited powers of vision among the remotest of the telescopic stars.

IV.

What is our little planet's day and night?
The day, that dazzle's human eyes with glare,
Checks not the interglance of starry light—
It beams incessantly forever there.

[The stars, except those near the sun, are at all times in very clear daylight mirrored in the limpid waters of deep wells; they are also visible at mid-day, reflected in the mountain lakes of northern latitudes, and especially when lofty and precipitous banks exclude all but the vertical light. The telescope, a chimney, or any opaque tube which entirely excludes the surrounding light, makes the stars visible from similar causes. When an observer, of a sunny day, shades his eye by placing his hand over it, it is to exclude the more direct rays, and thus to enable the less strong reflection of the distant landscape to be more visible. The tin tubes, blackened on the inside, which we use in looking at pictures, are useful for the same reason.

[The use of tubes, extending by uniting them to a great length, was known to Ptolemy and the ancients, but Metius, of Alcmaer in Holland, was the first who added glasses, thus inventing telescopes in 1609, which were successively improved by Galileo, Huygens, Newton, and others, to the time of Herschel.]

V.

We roll into Earth's shadow; and 'tis night
To us—our side the orb is from the sun;
One half's in shadow always—one in light;
Evening and morning since the world begun.

[One half of the earth's convex landscapes being always in light, and the other in shade, the earth's daily rotation on its axis rolls us into and out of its own shadow, causing day and night. The earth, rolling from West to East on its own axis, it is always mid-day and midnight at opposite meridians of longitude, and an eternal sunrise and sunset at the intermediate meridians.]

VI.

One half in day and summer light
The eternal changeful seasons turn;
One half in shade and winter night
The varying landscapes freeze and burn.

[The preceding stanza illustrates the relations of the sun's light caused by the earth's daily motion on its axis. This stanza refers to those changes which result from the earth's annual motion round the sun, and are caused by the inclination of the earth's axis to the plane of its annual path in its orbit, (the Ecliptic.) As only one half of the earth's convex landscapes can be in sunlight, it is obvious that day and night, and summer and winter, in all their varieties, must alternate constantly in the alternate hemispheres. As on a given parallel of latitude, that is on a line drawn East and West round the earth, mid-day, sunset, midnight, and sunrise, are perpetually present; so of a line drawn North and South round the earth, one half must be in sunlight, and the other in shade. There can be but one point where the sunlight falls vertically from the zenith, on the earth's surface. This point is always in the centre of the illumined portion of the earth, varying, however, constantly with the earth's revolution both on its axis and in its orbit round the sun. A spiral line is indicated by this point on the earth's surface, which, in three hundred and sixty-five days, has extended from Tropic to Tropic and back again, completing the course of the Seasons in both hemispheres. About six thousand miles, or one fourth of the earth's convex, from this vertical point, West, it is sunrise, six thousand East, it is sunset, and also six thousand miles North or South from this vertical ray, the sunlight must be falling parallel with the earth's surface on the icy deserts, both of the Arctic and Antarctic latitudes.

VII.

Myriads of sunbeams interweave their light,
Throughout the boundless distance of the sky;
And gem the spangled canopy of night,
Where'er the wanderer turns his thoughtful eye.

VIII.

The furthest starbeam's telescopic flight,
Direct, unerring from its centre runs;
Threads the vast maze of inter-radiant light
Athwart the day-source of a thousand suns.

IX.

On every side each distant sun displays,
Across the daylight of each other sun,

Its radiant sphere of still expanding rays,
  Widening and widening e'en since time begun.

X.

The feeblest starbeam in its furthest flight,
  Across the dazzling day-fires of the sky,
Truly reveals its centre and its source
  Unscorched and changeless to the gazer's eye.

XI.

E'en when the student swings great Herschel's lens,
  Measuring in mighty tracts concentric space;
Each telescopic sun his vision kens
  Further and feebler has its certain place.

XII.

In Cancer or in Leo, as we roll,
  One starry concave fills the midnight air;
Whether we view the Zodiac or the Pole,
  The constellated suns of space are there.

XIII.

As fades a stone's splash in the waves around,
  Though suns may darken at light's starting place;
While ages roll, and cycles wheel their bound,
  Light speeds, centrifugal, its onward race.

XIV.

Heaven's vast machine defies the optician's art,
  Naught but Omniscience its depths may scan,
What man may know is but a little part,
  All unrevealed to him the glorious plan.

XV.

Author of All! Almighty, yet unseen,
  Wondrous, surpassing wonder Thou must be,
Thou veil'st thyself beyond the starry scene,
  The light thou mad'st reveals thy works, not Thee.

XVI.

Thy Omnipresence shrouds itself in light,
Where its bright rays illume the furthest sky;
The tiny shadow of Earth's little night
Hides nothing from thine ever-seeing eye.

[The shadow of a planet, as of Venus, or of the earth, always exists on the side from the sun, and it is our rolling into and out of this shadow which causes day and night to us. It obstructs the whole sunlight no more than a raven's wing in the furthest horizon.

[The same laws of radiation from each point to every other point, the same wonderful inter-radiations are true of reflected light. Let us suppose a million of observers on a mountain side, above them the Heavens, before them a rich and varied landscape. Each observer, if he looks upward, receives on the expanse of the nerves of vision a daguerreotype, as it were, of the constellations. If he looks forward and around, he has a changing daguerreotype of the landscapes. The daguerreotype of the constellations may, it is true, be obscured by the daylight of our own star, but it is not the less certainly there.

[Provide each of these million of observers with telescopes of great power; each and all can now analyse the celestial area, and fix upon any one of the telescopic stars. Its feeblest light is present to all.

[Each and all in concert can next analyse the landscape, and reduce its vast variety to distinct vision. They can each direct their telescope to any given point in the field of vision. They can, with an engineer's common portable telescope, see the same man forty miles distant. Every point in the larger field of vision of the mountaineer, or smaller field of the citizen, bounded by the adjacent buildings, is really a centre of reflected light. The universe may be said to be filled with the direct radiant light from each of the fixed stars inter-radiating reciprocally with each of the other fixed stars. Through this the reflected light from the moon, and from each of the planets, and their satellites inter-radiates freely, as though it were the only light in nature. And also the radiation of light from every point of the distant landscape or the adjacent buildings is inter-radiating independently on all sides.

[The eye is the most delicate and accurate of optical instruments. The telescope lengthens the darkened vestibule of its camera-obscura-like chamber, and thus more effectively prevents the interference of lateral light. The most complex perspective, the most exquisite foliage of the distant landscape, which would be perceptible only by a telescope of a thousand powers, are unquestionably pictured with perfect exactness upon the eye of the rudest observer.

[As the measurement of all radiant phenomena, all forces acting from a centre explain and illustrate each other, any contrivance, whether pictorial, arithmetical, algebraic, or poetically descriptive, which explains one, aids in explaining all.]

THE BINOMIAL THEOREM.

The Cyclopædia of the Society for the Diffusion of Useful Knowledge, [London, 1838,] designates the Binomial Theorem as "by far the most important theorem of common Algebra. It was first announced by Newton, and perhaps may be con-

sidered as an invention, more than any other contribution of that philosopher. It is engraved on his tomb in Westminster Abbey.

The Encyclopedia Metropolitana, [London, 1829,] says, "Newton first assigned the law by which the binomial development was governed, but did not give any demonstration of it. Since his time, however, the theorem has been submitted to rigorous proof."

Rees' Cyclopedia, however, ascribes its invention to other authors; to Briggs and Stifelius, among the moderns, and also to Nicomachus and Bœthius, among the ancient Greeks.

In the course of this work we have already had occasion to reiterate that Algebra in every possible case is simply short hand; simply an abbreviated method of expressing truth in Arithmetic or in Geometry. The Algebra used may often appear very abstruse, and to the beginner, unless great care is taken, may often be a serious mystification of the subject it is designed to exemplify. But there is always a plain truth at the base of it. The beginner, before he can understand it, must learn the language in which it is expressed, just as he must learn the Greek language before he can interpret a Greek sentence.

As Sir Isaac's manner and intention was not directed especially to the simplification of truth to the plainest comprehension, we will now endeavor to make this famed and hitherto often formidable binomial theorem plain, intelligible, and significant to the scholar of the 19th century.

The Binomial Theorem is an example of the relation of Algebra to Geometry, of the relation which the simple truth bears to the formula by which it is expressed, and also is an excellent means of showing in the plainest and most tangible manner, the real difference between linear, superficial, and solid or cubical measurements.

Binomial from *bis* twice, and *nomen*, a name, means two names, two terms, two quantities. The binomial theorem signifies the measurement of two added quantities compared with the measurement of these same quantities separately. It may be easily understood by first considering the combinations of which measurements of simple linear, or one dimension are capable; secondly, those of which superficial, or two dimensions, and also of cubical or three dimensions are respectively capable.

Let A be any linear quantity whatever, say one inch, one yard, one mile. Let B be any linear quantity, say $\frac{1}{2}$ inch. The 1st power is algebraically stated thus:

$A + B$.

To explain: A added to B, is simply 1 linear inch added to $\frac{1}{2}$ inch linear, or $1\frac{1}{2}$ inches length.

The 2d power is algebraically

$(A + B)^2 = A^2 + 2(A \times B) + B^2$.

To explain: $(A+B)^2$ is A+B, multiplied by A+B, or $1\frac{1}{2}$ inches *length* multiplied by $1\frac{1}{2}$ inches *breadth*, which is thus exemplified in detail, or by retail analytically to the eye:

A×B	B^2
A^2	A×B

We have first A's square, that is, A's length multiplied into A's breadth, (A^2.) Secondly, on two sides of this square we have A's length multiplied by B's breadth. And thirdly B's length multiplied into B's breadth, forming the square of B, or B^2, completing the square of A+B. This would be invariably true in any possible variation of the two quantities. One is the two numbers added together and simply multiplied into each other, the other is an analysis, a dividing, a retailing of the same result.

Arithmetically, $1\frac{1}{2}$ multiplied by $1\frac{1}{2}$ is $2\frac{1}{4}$; of this A's square is 1. The two areas of A×B being each half of A's square = 1, and B's square is the $\frac{1}{4}$, Thus $1+1+\frac{1}{4}=2\frac{1}{4}$.

The 3d power is algebraically stated thus:

$(A + B)^3 = A^3 + 3(A^2 \times B) + 3(A \times B^2) + B^3$.

To explain: Take in hand the engraved cube of ivory or wood, which is intended to accompany this book.

It is $(A + B)^3$. That is A + B in length, multiplied by A + B in breadth, and this again multiplied by A + B in thickness gives the cube $(A + B)^3$, which is constituted of, and is capable of division as marked on the cube, first into A's length, breadth, and thickness, which is A's cube, (A^3.) Secondly, on three sides of this cube, we add A's length and breadth, (A^2) multiplied by B's thickness. Thirdly, in the corner diagonally opposite A's cube, we have B's length, breadth, and thickness, B's cube, (B^3.) Fourthly, on three sides of B's cube we have B's length and breadth, (B^2) multiplied by A's thickness, completing the cube of A+B.

Arithmetically, the mass of the whole cube in this example is $3\frac{3}{8}$. It is $1\frac{1}{2}\times 1\frac{1}{2}\times 1\frac{1}{2}=3\frac{3}{8}$, of which A's cube is 1. Secondly, on three sides of A's cube the three A's squares multiplied by or made up to B's thickness, are each one half of A's cube,$=1\frac{1}{2}$. Thirdly, on three sides of B's cube, the three B's squares multiplied by A's thickness, are each one quarter of A's cube, $=\frac{3}{4}$. Fourthly, B's cube, in the corner diagonally opposite A's cube, is one eighth of A's cube,$=\frac{1}{8}$, completing the cube. Thus $1+1\frac{1}{2}+\frac{3}{4}+\frac{1}{8}=3\frac{3}{8}$.

By using cubic inch blocks, and varying the quantities A and B at pleasure, the truth of the theorem will always be shown.

Thus far the exemplification of the binomial theorem well illustrates linear, superficial, and cubical measurements, the square and cube roots. Our primary elements, length,

breadth, and thickness are fundamental and universal; they are never to be lost sight of in any application of Algebra.

The 4th power is algebraically stated thus:

$(A+B)^4=A^4+4(A^3\times B)+6(A^2.\times B^2)+4(A\times B^3)+B^4$.

We might make a similar analysis of the 4th power, but it is not our present purpose, and would not compensate for the time. In any application to the laws of nature there is no 4th power, any more than there is a fortieth, a four hundredth, or a four thousandth power. After a proper understanding and amplification of the application of mathematical principles to the laws of nature, it might not be unadvisable for the student to occupy time enough to understand for himself that all powers beyond the 3d are but arithmetical and algebraic repetitions and permutations.

"Such mutations, like the diversity of languages may be deemed a serious evil, since they divert the attention to the mere accessaries of learning, and retard or obstruct the acquisition of real knowledge."—[*Encyclopedia Brittannica.*

THE TRINOMIAL.

$A\times C$	$B\times C$	C^2
$A\times B$	B^2	$B\times C$
A^2	$A\times B$	$A\times C$

The trinomial is a similar exemplification of the squares of three added linear quantities, compared with their squares separately considered.

The beginner who has first understood the binomial will easily perceive that we add to its exterior $2A\times C+2B\times C+C^2$.

We might refer to Plate

1, Fig 5, and, commencing with the first square, calling the first linear inch A, and the next B, C, and D, proceed to the quadrinomial, and so on indefinitely. If you start from the first square and build it up it is synthesis. If you commence from the larger quantity and divide it, it is analysis.

LOGARITHMS.

The labor and time required for performing the usual and ordinary arithmetical operations of multiplication, division, and the extraction of roots were serious obstacles to the progress of science, particularly the science of astronomy. Logarithms are a means of facilitating these calculations. Logarithms are a contrivance by which multiplication is made by addition, and division by subtraction. Tables of Logarithms are to the astronomer precisely what interest tables are to the bank clerk; they are tables constructed beforehand, which save him the labor of calculation in every individual case.

Logarithms may be defined to be the relation of the ratios of two series of numbers, *or quantities*, one *arithmetical*, and the other *geometrical*.

To explain: form two series of numbers, the one constituting a geometrical progression whose first term is unity or 1, and the common ratio any number whatever, and the other an arithmetical progression, whose first term is 0, and the common difference any number whatever. Thus:

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192.
0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13.

The terms of the arithmetical series are the logarithms of the corresponding terms of the geometrical series.

Now the addition or subtraction of any two of the lower

or arithmetical, produces the same result as the multiplication or division of the geometrical series. Thus, if we add 5+8=13, the number 8192, of which 13 is the logarithm, is the product of 32×256. Or, 12—3=9, the number 512 is the quotient of 4096 divided by 8.

Again, 1, 3, 9, 27, 81, 343, 1029, 3087, 9261, 27783, 83349,
0, 1, 2, 3, 4, 5, 6, 7, 8: 9, 10,

Logarithms, [from the Greek *logos, used in the sense of* a ratio, and *arithmos*, a number,] are tables exhibiting arithmetically the relation of roots and of their powers; the geometrical relations of length, breadth and thickness; the relations of circles or of spheres to their respective radii; the relations of squares and of cubes to their respective roots. They were invented in 1614, by John Napier, Baron of Merchiston, Scotland.

The simplest notion which can be given of logarithms is to consider them as representing a series of simple linear quantities, in comparison with superficial or solid quantities. Thus in Plate No. 8, and in Plate No. 17, the simple linear quantities are the sides of the respective squares, while the areas of the squares and circles represent the powers, in relation to motion; and if the areas were to represent cubes they would represent the intensity of radiating forces.

Logarithms, binomials, and all other geometrical considerations will be found to depend on a few grand and simple principles. Logarithms are but arithmetical statements of geometrical ratios.

ALGEBRA.

We have defined (see pages 34, and 42 to 47,) mathematics to be *measurement*, and that it is divided into Arithmetic, Geometry, and Algebra, of which the last is but short hand.

In defining Algebra to be a method of short hand, and in attempting to present more clearly the truths which form the basis of Algebraic formulas, we have no intention to underrate its intrinsic and real value. As an additional evidence that the existing methods are not satisfactory, the following quotations are copied from a work of the latest and highest authority:

"In short, if Algebra were purged of the vicious language and inaccurate conceptions which pervade it, and which were early introduced by the confidence of rapid and careless calculators, it might at last claim the character of a perfect science.

"The number of elementary treatises on Algebra which have been produced in various languages during the currency of the eighteenth century, appears quite incredible. They are almost ephemeral, and assume every feature and dimension, superficial or profound, from the size of a primer to the magnitude of a lexicon.

"Nothing is more wanted for the purpose of education than a classical treatise on Algebra, which, avoiding all vague terms and hasty analogies, should unfold the principles with simplicity and rigid accuracy, and follow the train of induction with close and philosophical circumspection."

[*Sir John Leslie, Encyclopedia Brittannica, 7th edition.*

CONCLUDING REMARKS.

Our design to aid the student in forming a general conception of the great truths which it is the purpose of mathematics to unravel, and to identify the three varieties of measurement, linear, superficial and massive, and their relation to motion, rectilinear, curvilinear and [illegible]iant, may now be to some extent recognised. The subject is too comprehensive to be easily exhausted, but the author feels that he must bring this volume to an abrupt conclusion, asking that it be read connectedly, with some indulgence for imperfections incident to a first edition.

It is not improbable that some important extension of the work is essential to its completeness; but brevity is the soul of wit, and the present volume is submitted with this merit to the judgment of the public. The author hopes that fewer imperfections may be found in succeeding editions.